Algebraic Bethe ansatz for integrable one-dimensional extended Hubbard models with open boundary conditions

2003 J. Phys. A: Math. Gen. 36 1801

(http://iopscience.iop.org/0305-4470/36/7/302)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 1801–1825 PII: S0305-4470(03)52680-4

Algebraic Bethe ansatz for integrable one-dimensional extended Hubbard models with open boundary conditions

Xiang-Yu Ge and Mark D Gould

Centre for Mathematical Physics, The University of Queensland, Brisbane, Qld 4072, Australia

E-mail: xg@maths.uq.edu.au

Received 22 August 2002, in final form 25 November 2002
Published 5 February 2003
Online at stacks.iop.org/JPhysA/36/1801

Abstract
Nine classes of integrable open boundary conditions, further extending the one-dimensional $U_q(gl(2|2))$ extended Hubbard model, have been constructed previously by means of the boundary $\mathbf{Z}_2$-graded quantum inverse scattering method. The boundary systems are now solved by using the algebraic Bethe ansatz method, and the Bethe ansatz equations are obtained for all nine cases.

PACS numbers: 71.20.Fd, 75.10.Jm, 75.10.Lp

1. Introduction

It is well known that exact solutions of one-dimensional strongly correlated electron models can be obtained by means of the Bethe ansatz method, which was first introduced in the famous paper by Bethe [1] in 1931 on the one-dimensional Heisenberg model of a nearest-neighbour interaction, isotropic spin-$\frac{1}{2}$ chain. The first electron model solved exactly by Bethe's method was the nonrelativistic continuum model of electrons with local interactions, treated by Yang [2] in 1967. The study of the exact solution of the one-dimensional Hubbard model starts with the paper of Lieb and Wu [3] in 1968, where the equations now known as the coordinate Bethe ansatz or Lieb–Wu equations were derived. The coordinate Bethe anstaz method was developed further by Baxter, who solved the XYZ Heisenberg model [4] using what is now called the Yang–Baxter equation. The one-dimensional extended Hubbard model corresponds to the integrable Hamiltonians associated with the Lie algebras $gl(2|1)$ and $gl(2|2)$, as derived by Uimin, Lai and Sutherland (ULS) [5–7] in the 1970s. A great impulse to the theory of integrable systems was given by Takhtajan *et al* [8], who proposed the quantum inverse scattering method (QISM) or the algebraic Bethe ansatz, to synthesize the treatment of integrable quantum systems.

Since the discovery of high-T_c superconductivity, there has been continuing interest in lattice models of strongly correlated electrons. This is especially so for models that are

0305-4470/03/071801+25$30.00

thought to capture some of the features of superconductivity, while being integrable or exactly solvable. The Hubbard model and the $t - J$ model have attracted a great deal of attention, initiated by Anderson [9] and Zhang and Rice [10], in relation to high-T_c superconductivity. For the one-dimensional Hubbard model, the QISM provides us with more information on algebraic aspects. The R-matrix which satisfies the Yang–Baxter relation was found and the integrability of the model was established by Shastry [11]. Woynarovich [12] applied the finite-size correction of the ground state energy to obtain the low-lying gapless excitation spectrum around the ground state. The critical exponents of the various correlation functions were subsequently obtained by Frahm and Korepin [13], and the algebraic Bethe ansatz approach to the derivation of the eigenstates and the eigenvalues of the transfer matrix using the R-matrix was reported by Ramos and Martins [14].

The $t - J$ model with special coupling constants ($J = \pm 2t$) was introduced and solved by means of the nested Bethe ansatz method by Lai [6] in 1974 and Sutherland [7] in 1975. The two different forms of the Bethe ansatz equations are referred to as the Lai and Sutherland representations, respectively. In 1988, the $t - J$ model, as one of the simplest models for studying strongly correlated electron systems, was proposed [10] as a subcase of the Hubbard model. In 1990, Bares *et al* [15] and Sarkar [16] independently discovered the supersymmetry of the model. They applied the supersymmetry to construct the Bethe ansatz solution in Sutherland's form, and discussed the ground state and low-lying excitations close to half-filling. A third form of representation of the Bethe ansatz was discovered by Essler and Korepin [17] in 1992, who established the integrability of the model in the framework of the graded QISM. Then a $U_q(gl(2|1))$ invariant supersymmetric $t - J$ model was proposed and discussed by Klümper *et al* [18], Bariev [19] and Forerster *et al* [20].

A variety of exactly solvable one-dimensional lattice models of strongly correlated electrons have been proposed subsequently, including the so-called supersymmetric Essler–Korepin–Schoutens (EKS) extended Hubbard model [21], which was proposed by Essler *et al* in 1992, and the so-called supersymmetric U model, proposed by Bracken *et al* [22] in 1995 and extensively investigated subsequently by Bedürftig and Frahm [23], Pfannmüller and Frahm [24] and Ramos and Martins [25]. The EKS model is a $gl(2|2)$ supersymmetric model containing the supersymmetric $t - J$ model as a submodel, and can be interpreted as the Hubbard model plus moderate nearest-neighbour interactions. The complete solution of the EKS model by the algebraic Bethe ansatz has been extensively investigated [26]. The mathematical question of the completeness of the solution has been settled by Schoutens [27], and the physical content of the solution, in particular the low-lying excitations, has been studied by Essler and Korepin [28]. Of particular importance is the fact that the solution exhibits so-called off-diagonal long-range order, of relevance to superconductivity. Physical applications of integrable spin chains associated with $gl(m|n)$ were discussed in a review paper by Shlottmann [11] and more recently by Ambjøm *et al* [30].

One of the most important developments in the theory of completely integrable lattice models with open boundary conditions is Sklyanin's work [31] in 1988 on the boundary QISM, which originated in previous work on reflection equations by Cherednik [32] in 1984. The important ingredient in this boundary QISM is a new algebraic structure, the reflection equation algebra. Sklyanin used his formalism to solve, using the algebraic Bethe ansatz method, the spin-$\frac{1}{2}$ open Heisenberg XXZ chain with boundary terms, which had already been solved via the coordinate Bethe ansatz method by Alcaraz *et al* [33]. In 1991, Mezincescu and Nepomechie [34] extended the results of Sklyanin to the case of R-matrices which do not satisfy separate P-invariance or T-invariance, in addition to crossing symmetry, but only the less restrictive condition of the combined PT-invariance.

Open boundary conditions and boundary fields for one-dimensional Hubbard model have been studied by Zhou [35], and by Shiroishi and Wadati [36], the finite-size corrections to the Hubbard model with boundary fields have been presented by Asakawa and Suzuki [37], Shiroishi and Wadati [36] and by Tsuchiya and Yamamoto [38], some physical quantities such as the spectrum of boundary states and the magnetization at the edge site have been evaluated by Bedürftig and Frahm [39] and by Yue and Deguch [40], then the Bethe ansatz equations were obtained, using the algebraic Bethe ansatz method, by Guan [41]. Integrable open boundary conditions for the supersymmetric $t-J$ model with the $U_q(gl(2|1))$ quantum group invariance were proposed by Foerster and Karowski [20] and by González-Ruiz [42]. The zero-temperature boundary effects in an open $t-J$ model with boundary fields were discussed by Essler [43], the finite-size corrections to the supersymmetric $t-J$ model with boundary fields were presented by Asakawa and Suzuki [44], and some exact results about the physical aspects of the $t-J$ model with the $U_q(gl(2|1))$ symmetry were extracted by Bariev *et al* [46]. Recently in the work of Bracken *et al* [47], we extended Sklyanin's work to the supersymmetric case, proposing a general graded reflection equation algebra and formulating the corresponding graded boundary QISM. There are several papers in the work of Zhou *et al* [48, 49] and Fan *et al* [50], where the construction of the eigenvectors is discussed by extending Sklyanin's open boundary algebraic Bethe ansatz and the nested algebraic Bethe anstaz for the supersymmetric $t-J$ model with $U_q(gl(2|1))$ symmetry, and the construction is closely related to the present paper.

In [51], boundary conditions and non-trivial boundary interactions for the q-deformed EKS model, which are compatible with integrability in the bulk, have been constructed from solutions of the graded reflection equations in the framework of the graded boundary QISM [47]. The purpose of the present work is to solve this extended q-deformed EKS model [51] by means of the algebraic Bethe ansatz, and to derive the associated Bethe ansatz equations.

2. The Hamiltonian with open boundary conditions

Let $c_{j,\sigma}^\dagger$ and $c_{j,\sigma}$ denote creation and annihilation operators for conduction electrons with spin σ at site j, satisfying the anti-commutation relations given by $\{c_{i,\sigma}^\dagger, c_{j,\sigma'}\} = \delta_{ij}\delta_{\sigma\sigma'}$, where $i, j = 1, 2, \ldots, L$ and $\sigma, \sigma' = \uparrow, \downarrow$. At a given lattice site j there are four possible electronic states,

$$|0\rangle \qquad |\uparrow\rangle_j = c_{j,\uparrow}^\dagger|0\rangle \qquad |\downarrow\rangle_j = c_{j,\downarrow}^\dagger|0\rangle \qquad |\uparrow,\downarrow\rangle_j = c_{j,\downarrow}^\dagger c_{j,\uparrow}^\dagger|0\rangle.$$

We consider the following type of Hamiltonian, describing the open boundary conditions for the q-deformed EKS model [21, 51] on a one-dimensional lattice:

$$\begin{aligned} H = &-\sum_{j=1}^{L-1}\sum_{\sigma}\left(c_{j,\sigma}^\dagger c_{j+1,\sigma} + \text{H.c.}\right)(1 - n_{j,-\sigma} - n_{j+1,-\sigma}) \\ &\qquad - \sum_{j=1}^{L-1}\left(c_{j,\uparrow}^\dagger c_{j,\downarrow}^\dagger c_{j+1,\downarrow} c_{j+1,\uparrow} - S_j^+ S_{j+1}^- + \text{H.c.}\right) \\ &\qquad + \mathrm{e}^{-\gamma}\sum_{j=1}^{L-1}(n_j - n_{j,\uparrow}n_{j,\downarrow} + n_{j,\downarrow}n_{j+1,\uparrow}) \\ &\qquad + \mathrm{e}^{\gamma}\sum_{j=1}^{L-1}(n_{j+1} - n_{j+1,\uparrow}n_{j+1,\downarrow} + n_{j,\uparrow}n_{j+1,\downarrow}) \end{aligned}$$

$$+4\cosh\gamma\sum_{j=1}^{L-1}\left(S_j^z S_{j+1}^z-\frac{1}{4}n_j n_{j+1}\right)$$
$$+2\sinh\gamma\sum_{j=1}^{L-1}(n_{j,\uparrow}n_{j,\downarrow}n_{j+1,\uparrow}-n_{j,\uparrow}n_{j+1,\uparrow}n_{j+1,\downarrow})+H_{lt}^{\text{boundary}}+H_{rt}^{\text{boundary}}. \qquad (1)$$

In the above, S_j^+, S_j^-, S_j^z as usual are the vector spin operators for the conduction electrons at site j satisfying the $su(2)$ algebra and expressed as $S_j^+=c_{j,\uparrow}^\dagger c_{j,\downarrow}$, $S_j^-=c_{j,\downarrow}^\dagger c_{j,\uparrow}$, $S_j^z=\frac{1}{2}(n_{j,\uparrow}-n_{j,\downarrow})$; $\sigma_g^\pm=\sigma_g^x\pm\mathrm{i}\sigma_g^y$, σ_g^z $(g=a,b)$ are the local moments with spin-$\frac{1}{2}$ located at the left and right ends of the system respectively.

We propose the following nine classes of boundary conditions, obtained by simple transformation of those given in [51]:

$$\text{Case (i):}\quad H_{lt}^{\text{boundary}}=\mathrm{e}^{-\gamma}\left(\left(1-\frac{\mathrm{e}^{\gamma(1+\frac{\xi_-^I}{2})}\sinh\gamma}{\sinh\frac{\gamma\xi_-^I}{2}}\right)n_{1\uparrow}n_{1\downarrow}-n_1\right)$$
$$H_{rt}^{\text{boundary}}=\mathrm{e}^{\gamma}\left(\left(1-\frac{\mathrm{e}^{-\gamma(1+\frac{\xi_+^I}{2})}\sinh\gamma}{\sinh\frac{\gamma\xi_+^I}{2}}\right)n_{L\uparrow}n_{L\downarrow}-n_L\right) \qquad (2)$$

$$\text{Case (ii):}\quad H_{lt}^{\text{boundary}}=-\mathrm{e}^{-\gamma}\left(\frac{\mathrm{e}^{\gamma}\sinh\gamma\left(\frac{\xi_-^{II}}{2}+1\right)}{\sinh\frac{\gamma\xi_-^{II}}{2}}n_{1\uparrow}+n_{1\downarrow}-n_{1\uparrow}n_{1\downarrow}\right)$$
$$H_{rt}^{\text{boundary}}=-\mathrm{e}^{\gamma}\left(\frac{\mathrm{e}^{-\gamma}\sinh\gamma\left(\frac{\xi_+^{II}}{2}+1\right)}{\sinh\frac{\gamma\xi_+^{II}}{2}}n_{L\uparrow}+n_{L\downarrow}-n_{L\uparrow}n_{L\downarrow}\right) \qquad (3)$$

$$\text{Case (iii):}\quad H_{lt}^{\text{boundary}}=-\frac{\sinh\gamma\left(1+\frac{\xi_-^{III}}{2}\right)}{\sinh\frac{\gamma\xi_-^{III}}{2}}\left(n_1-\left(1+\frac{\mathrm{e}^{-\frac{\gamma\xi_-^{III}}{2}}\sinh\gamma}{\sinh\gamma\left(1+\frac{\xi_-^{III}}{2}\right)}\right)n_{1\uparrow}n_{1\downarrow}\right)$$
$$H_{rt}^{\text{boundary}}=-\frac{\sinh\gamma\left(1+\frac{\xi_+^{III}}{2}\right)}{\sinh\frac{\gamma\xi_+^{III}}{2}}\left(n_L-\left(1+\frac{\mathrm{e}^{\frac{\gamma\xi_+^{III}}{2}}\sinh\gamma}{\sinh\gamma\left(1+\frac{\xi_+^{III}}{2}\right)}\right)n_{L\uparrow}n_{L\downarrow}\right) \qquad (4)$$

$$\text{Case (iv):}\quad H_{lt}^{\text{boundary}}=\mathrm{e}^{-\gamma}\left(\left(1-\frac{\mathrm{e}^{\gamma(1+\frac{\xi_-^I}{2})}\sinh\gamma}{\sinh\frac{\gamma\xi_-^I}{2}}\right)n_{1\uparrow}n_{1\downarrow}-n_1\right)$$
$$H_{rt}^{\text{boundary}}=-\mathrm{e}^{\gamma}\left(\frac{\mathrm{e}^{-\gamma}\sinh\gamma\left(\frac{\xi_+^{II}}{2}+1\right)}{\sinh\frac{\gamma\xi_+^{II}}{2}}n_{L\uparrow}+n_{L\downarrow}-n_{L\uparrow}n_{L\downarrow}\right) \qquad (5)$$

$$\text{Case (v):}\quad H_{lt}^{\text{boundary}}=-\mathrm{e}^{-\gamma}\left(\frac{\mathrm{e}^{\gamma}\sinh\gamma\left(\frac{\xi_-^{II}}{2}+1\right)}{\sinh\frac{\gamma\xi_-^{II}}{2}}n_{1\uparrow}+n_{1\downarrow}-n_{1\uparrow}n_{1\downarrow}\right)$$
$$H_{rt}^{\text{boundary}}=\mathrm{e}^{\gamma}\left(\left(1-\frac{\mathrm{e}^{-\gamma(1+\frac{\xi_+^I}{2})}\sinh\gamma}{\sinh\frac{\gamma\xi_+^I}{2}}\right)n_{L\uparrow}n_{L\downarrow}-n_L\right) \qquad (6)$$

$$
\begin{aligned}
\text{Case (vi):}\quad H_{lt}^{\text{boundary}} &= \mathrm{e}^{-\gamma}\left(\left(1-\frac{\mathrm{e}^{\gamma(1+\frac{\xi_-^{I}}{2})}\sinh\gamma}{\sinh\frac{\gamma\xi_-^{I}}{2}}\right)n_{1\uparrow}n_{1\downarrow}-n_1\right)\\
H_{rt}^{\text{boundary}} &= -\frac{\sinh\gamma\left(1+\frac{\xi_+^{III}}{2}\right)}{\sinh\frac{\gamma\xi_+^{III}}{2}}\left(n_L-\left(1+\frac{\mathrm{e}^{\frac{\gamma\xi_+^{III}}{2}}\sinh\gamma}{\sinh\gamma\left(1+\frac{\xi_+^{III}}{2}\right)}\right)n_{L\uparrow}n_{L\downarrow}\right)
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\text{Case (vii):}\quad H_{lt}^{\text{boundary}} &= -\frac{\sinh\gamma\left(1+\frac{\xi_-^{III}}{2}\right)}{\sinh\frac{\gamma\xi_-^{III}}{2}}\left(n_1-\left(1+\frac{\mathrm{e}^{-\frac{\gamma\xi_-^{III}}{2}}\sinh\gamma}{\sinh\gamma\left(1+\frac{\xi_-^{III}}{2}\right)}\right)n_{1\uparrow}n_{1\downarrow}\right)\\
H_{rt}^{\text{boundary}} &= \mathrm{e}^{\gamma}\left(\left(1-\frac{\mathrm{e}^{-\gamma(1+\frac{\xi_+^{I}}{2})}\sinh\gamma}{\sinh\frac{\gamma\xi_+^{I}}{2}}\right)n_{L\uparrow}n_{L\downarrow}-n_L\right)
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\text{Case (viii):}\quad H_{lt}^{\text{boundary}} &= -\mathrm{e}^{-\gamma}\left(\frac{\mathrm{e}^{\gamma}\sinh\gamma\left(\frac{\xi_-^{II}}{2}+1\right)}{\sinh\frac{\gamma\xi_-^{II}}{2}}n_{1\uparrow}+n_{1\downarrow}-n_{1\uparrow}n_{1\downarrow}\right)\\
H_{rt}^{\text{boundary}} &= -\frac{\sinh\gamma\left(1+\frac{\xi_+^{III}}{2}\right)}{\sinh\frac{\gamma\xi_+^{III}}{2}}\left(n_L-\left(1+\frac{\mathrm{e}^{\frac{\gamma\xi_+^{III}}{2}}\sinh\gamma}{\sinh\gamma\left(1+\frac{\xi_+^{III}}{2}\right)}\right)n_{L\uparrow}n_{L\downarrow}\right)
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\text{Case (ix):}\quad H_{lt}^{\text{boundary}} &= -\frac{\sinh\gamma\left(1+\frac{\xi_-^{III}}{2}\right)}{\sinh\frac{\gamma\xi_-^{III}}{2}}\left(n_1-\left(1+\frac{\mathrm{e}^{-\frac{\gamma\xi_-^{III}}{2}}\sinh\gamma}{\sinh\gamma\left(1+\frac{\xi_-^{III}}{2}\right)}\right)n_{1\uparrow}n_{1\downarrow}\right)\\
H_{rt}^{\text{boundary}} &= -\mathrm{e}^{\gamma}\left(\frac{\mathrm{e}^{-\gamma}\sinh\gamma\left(\frac{\xi_+^{II}}{2}+1\right)}{\sinh\frac{\gamma\xi_+^{II}}{2}}n_{L\uparrow}+n_{L\downarrow}-n_{L\uparrow}n_{L\downarrow}\right).
\end{aligned}
\tag{10}
$$

It has been shown [51] that the bulk Hamiltonian acquires an underlying supersymmetry algebra $U_q(gl(2|2))$ in the minimal representation. Furthermore, open chain integrability with appropriate boundary conditions was established using the boundary QISM. Let us recall that the local Hamiltonian of the supersymmetric q-deformed extended Hubbard model is derived from an R-matrix, satisfying the Yang–Baxter equation, which has the form [51]

$$
R(u)=\begin{pmatrix}
a(u)&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&b(u)&0&0&c(u)&0&0&0&0&0&0&0&0&0&0&0\\
0&0&b(u)&0&0&0&0&0&c(u)&0&0&0&0&0&0&0\\
0&0&0&b(u)&0&0&0&0&0&0&0&0&c(u)&0&0&0\\
0&d(u)&0&0&b(u)&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&a(u)&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&b(u)&0&0&c(u)&0&0&0&0&0&0\\
0&0&0&0&0&0&0&b(u)&0&0&0&0&0&c(u)&0&0\\
0&0&d(u)&0&0&0&0&0&b(u)&0&0&0&0&0&0&0\\
0&0&0&0&0&0&d(u)&0&0&b(u)&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&e(u)&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&b(u)&0&0&-c(u)&0\\
0&0&0&d(u)&0&0&0&0&0&0&0&0&b(u)&0&0&0\\
0&0&0&0&0&0&0&d(u)&0&0&0&0&0&b(u)&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-d(u)&0&0&b(u)&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&e(u)
\end{pmatrix}
\tag{11}
$$

$$a(u)=1 \quad b(u)=\frac{\sinh\frac{\gamma u}{2}}{\sinh\frac{\gamma(u+\eta)}{2}} \quad c(u)=\frac{\sinh\frac{\gamma u}{2}\,\mathrm{e}^{-\frac{\gamma u}{2}}}{\sinh\frac{\gamma(u+\eta)}{2}}$$
$$d(u)=\frac{\sinh\frac{\gamma u}{2}\,\mathrm{e}^{\frac{\gamma u}{2}}}{\sinh\frac{\gamma(u+\eta)}{2}} \quad e(u)=\frac{\sinh\frac{\gamma(u-\eta)}{2}}{\sinh\frac{\gamma(u+\eta)}{2}}$$

where u is the spectral parameter.

We remark that the four possible electronic states of the vector irrespective of the quantum superalgebra $U_q(gl(2|2))$ symmetry are identified with

$$|1\rangle=|0\rangle \qquad |2\rangle=c^{\dagger}_{j,\downarrow}c^{\dagger}_{j,\uparrow}|0\rangle \qquad |3\rangle=c^{\dagger}_{j,\uparrow}|0\rangle \qquad |4\rangle=c^{\dagger}_{j,\downarrow}|0\rangle \tag{12}$$

and we choose to adopt the bosonic, bosonic, fermionic and fermionic (BBFF) grading $[|1\rangle]=[|2\rangle]=0$, $[|3\rangle]=[|4\rangle]=1$ on the indices labelling the basis vectors.

3. The Bethe ansatz equations

3.1. Boundary K-matrices

The quantum integrability of the system defined by the Hamiltonian (1) with any of the nine classes of open boundary conditions (2)–(10) can be established by means of the graded boundary QISM. We first search for the diagonal boundary K-matrices $K_{\pm}(u)$ which satisfy the graded reflection equation

$$R_{12}(u_1-u_2)\overset{1}{\mathcal{T}}_-(u_1)R_{21}(u_1+u_2)\overset{2}{\mathcal{T}}_-(u_2)=\overset{2}{\mathcal{T}}_-(u_2)R_{12}(u_1+u_2)\overset{1}{\mathcal{T}}_-(u_1)R_{21}(u_1-u_2) \tag{13}$$

and the dual graded reflection equation

$$\begin{aligned}R_{21}^{st_1ist_2}(-u_1+u_2)\overset{1}{\mathcal{T}}{}_+^{st_1}(u_1)\left\{\left[R_{21}^{st_1}(u_1+u_2)\right]^{-1}\right\}^{ist_2}\overset{2}{\mathcal{T}}{}_+^{ist_2}(u_2)\\ =\overset{2}{\mathcal{T}}{}_+^{ist_2}(u_2)\left\{\left[R_{12}^{ist_2}(u_1+u_2)\right]^{-1}\right\}^{st_1}\overset{1}{\mathcal{T}}{}_+^{st_1}(u_1)R_{12}^{st_1ist_2}(-u_1+u_2).\end{aligned} \tag{14}$$

Here the supertransposition st_α ($\alpha=1,2$) is only carried out in the αth factor superspace of $V\otimes V$, whereas ist_α denotes the inverse operation of st_α. The two associative superalgebras $\mathcal{T}_-$ and $\mathcal{T}_+$ are defined by the R-matrix of the model, $R(u)=R_{12}(u)$ as in (13) and $R_{21}(u)=P_{12}R_{12}(u)P_{12}$ with P being the $\mathbf{Z}_2$-graded permutation operator, with BBFF grading.

By modifying Sklyanin's arguments [47], one may show that the quantities $\tau(u)$ given by $\tau(u)=\mathrm{str}(\mathcal{T}_+(u)\mathcal{T}_-(u))$ constitute a commutative family, i.e. $[\tau(u_1),\tau(u_2)]=0$.

After complicated algebraic manipulations, we find three different diagonal boundary K-matrices, $K_-^I(u)$, $K_-^{II}(u)$ and $K_-^{III}(u)$, which solve the graded reflection equation (13):

$$\begin{aligned}K_-^I(u)=\frac{1}{\sinh\frac{\gamma\xi_-^I}{2}}\,\mathrm{diag}\bigg(&\sinh\frac{\gamma(u+\xi_-^I)}{2},\mathrm{e}^{\gamma u}\sinh\frac{\gamma(-u+\xi_-^I)}{2},\\ &\mathrm{e}^{2\gamma u}\sinh\frac{\gamma(u+\xi_-^I)}{2},\mathrm{e}^{2\gamma u}\sinh\frac{\gamma(u+\xi_-^I)}{2}\bigg)\\ K_-^{II}(u)=\frac{1}{\sinh\frac{\gamma\xi_-^{II}}{2}}\,\mathrm{diag}\bigg(&\sinh\frac{\gamma(u+\xi_-^{II})}{2},\mathrm{e}^{\gamma u}\sinh\frac{\gamma(-u+\xi_-^{II})}{2},\\ &\mathrm{e}^{\gamma u}\sinh\frac{\gamma(-u+\xi_-^{II})}{2},\mathrm{e}^{2\gamma u}\sinh\frac{\gamma(u+\xi_-^{II})}{2}\bigg)\end{aligned}$$

$$K_-^{III}(u) = \frac{1}{\sinh\frac{\gamma\xi_-^{III}}{2}}\,\mathrm{diag}\left(\sinh\frac{\gamma(u+\xi_-^{III})}{2}, \sinh\frac{\gamma(u+\xi_-^{III})}{2},\right.$$
$$\left. \mathrm{e}^{\gamma u}\sinh\frac{\gamma(-u+\xi_-^{III})}{2}, \mathrm{e}^{\gamma u}\sinh\frac{\gamma(-u+\xi_-^{III})}{2}\right). \tag{15}$$

The corresponding diagonal boundary K-matrices, $K_+^{I}(u)$, $K_+^{II}(u)$ and $K_+^{III}(u)$, which obey the dual graded reflection equation (14), can be derived by isomorphism,

$$K_+^{I,II,III}(u) = MK_-^{I,II,III}(-u)$$

where $M = \mathrm{diag}(1, \mathrm{e}^{-2\gamma}, \mathrm{e}^{-2\gamma}, 1)$ is a crossing matrix. Therefore, we can choose three different diagonal boundary K-matrices $K_+^{I}(u)$, $K_+^{II}(u)$ and $K_+^{III}(u)$ as

$$K_+^{I}(u) = \frac{1}{\sinh\frac{\gamma(\xi_+^{I}-2)}{2}}\,\mathrm{diag}\left(\sinh\frac{\gamma\left(u+\xi_+^{I}-2\right)}{2}, \mathrm{e}^{-\gamma(u+2)}\sinh\frac{\gamma\left(-u+\xi_+^{I}-2\right)}{2},\right.$$
$$\left.\mathrm{e}^{-\gamma(2u+2)}\sinh\frac{\gamma\left(u+\xi_+^{I}-2\right)}{2}, \mathrm{e}^{-2\gamma u}\sinh\frac{\gamma\left(u+\xi_+^{I}-2\right)}{2}\right)$$
$$K_+^{II}(u) = \frac{1}{\sinh\frac{\gamma\xi_+^{II}}{2}}\,\mathrm{diag}\left(\sinh\frac{\gamma\left(u+\xi_+^{II}\right)}{2}, \mathrm{e}^{-\gamma(u+2)}\sinh\frac{\gamma\left(-u+\xi_+^{II}\right)}{2},\right.$$
$$\left.\mathrm{e}^{-\gamma(u+2)}\sinh\frac{\gamma\left(-u+\xi_+^{II}\right)}{2}, \mathrm{e}^{-2\gamma u}\sinh\frac{\gamma\left(u+\xi_+^{II}\right)}{2}\right) \tag{16}$$
$$K_+^{III}(u) = \frac{1}{\sinh\frac{\gamma\xi_+^{III}}{2}}\,\mathrm{diag}\left(\sinh\frac{\gamma\left(u+\xi_+^{III}\right)}{2}, \mathrm{e}^{-2\gamma}\sinh\frac{\gamma\left(u+\xi_+^{III}\right)}{2},\right.$$
$$\left.\mathrm{e}^{-\gamma(u+2)}\sinh\frac{\gamma\left(-u+\xi_+^{III}\right)}{2}, \mathrm{e}^{-\gamma u}\sinh\frac{\gamma\left(-u+\xi_+^{III}\right)}{2}\right).$$

3.2. The first-level algebraic Bethe ansatz

Having established the quantum integrability of the models, let us first diagonalize the Hamiltonian (1) by means of the algebraic Bethe ansatz method [31, 47]. We write the solution $K_+(u)$ of the dual graded reflection equation (14), and the 'doubled' monodromy matrix $\mathcal{T}(u)$, in the following forms:

$$K_+(u) = \mathrm{diag}\left(k_1^+(u), k_2^+(u), k_3^+(u), k_4^+(u)\right)$$
$$\mathcal{T}(u) = T(u)K_-(u)T^{-1}(-u) \equiv \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}_1(u) & \mathcal{B}_2(u) & \mathcal{B}_3(u) \\ \mathcal{C}_1(u) & \mathcal{D}_{11}(u) & \mathcal{D}_{12}(u) & \mathcal{D}_{13}(u) \\ \mathcal{C}_2(u) & \mathcal{D}_{21}(u) & \mathcal{D}_{22}(u) & \mathcal{D}_{23}(u) \\ \mathcal{C}_3(u) & \mathcal{D}_{31}(u) & \mathcal{D}_{32}(u) & \mathcal{D}_{33}(u) \end{pmatrix}. \tag{17}$$

In order to obtain the commutation relations, we need the following transformation:

$$\mathcal{D}_{bd}(u) = \check{\mathcal{D}}_{bd}(u) + \frac{\mathrm{e}^{\gamma u}\sinh\frac{\gamma\eta}{2}}{\sinh\frac{\gamma(2u+\eta)}{2}}\delta_{bd}\mathcal{A}(u).$$

Because $\mathcal{T}(u)$ satisfies the reflection equation (13), we obtain the following commutation relations:

$$\check{\mathcal{D}}_{bd}(u)\mathcal{B}_c(v) = \frac{\sinh\frac{\gamma(u-v-2)}{2}\sinh\frac{\gamma(u+v-4)}{2}}{\sinh\frac{\gamma(u-v)}{2}\sinh\frac{\gamma(u+v-2)}{2}} r(u+v-2)^{eb}_{gh} r(u-v)^{ih}_{cd}\mathcal{B}_e(v)\check{\mathcal{D}}_{gi}(u)$$
$$- \frac{e^{\frac{\gamma(u+v)}{2}}\sinh\gamma\sinh\gamma(u-2)\sinh\gamma v}{\sinh\frac{\gamma(u+v-2)}{2}\sinh\gamma(u-1)\sinh\gamma(v-1)} r(2u-2)^{gb}_{cd}\mathcal{B}_g(u)\mathcal{A}(v)$$
$$+ \frac{e^{\frac{\gamma(u-v)}{2}}\sinh\gamma\sinh\gamma(u-2)}{\sinh\frac{\gamma(u-v)}{2}\sinh\gamma(u-1)} r(2u-2)^{gb}_{id}\mathcal{B}_g(u)\check{\mathcal{D}}_{ic}(v) \tag{18}$$

$$\mathcal{A}(u_1)\mathcal{B}_b(u_2) = \frac{\sinh\frac{\gamma(u-v+2)}{2}\sinh\frac{\gamma(u+v)}{2}}{\sinh\frac{\gamma(u-v)}{2}\sinh\frac{\gamma(u+v-2)}{2}}\mathcal{B}_b(v)\mathcal{A}(u) + \frac{e^{-\frac{\gamma(u-v)}{2}}\sinh\gamma\sinh\gamma v}{\sinh\gamma(v-1)\sinh\frac{\gamma(u-v)}{2}}\mathcal{B}_b(u)\mathcal{A}(v)$$
$$+ \frac{e^{-\frac{\gamma(u+v)}{2}}\sinh\gamma}{\sinh\frac{\gamma(u+v-2)}{2}}\mathcal{B}_d(u)\check{\mathcal{D}}_{db}(v). \tag{19}$$

Here the indices take values 1, 2, 3 and the r-matrix is defined as

$$r(u) = \begin{pmatrix} a(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(u) & 0 & c(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b(u) & 0 & 0 & 0 & c(u) & 0 & 0 \\ 0 & d(u) & 0 & b(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e(u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(u) & 0 & -c(u) & 0 \\ 0 & 0 & d(u) & 0 & 0 & 0 & b(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -d(u) & 0 & b(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e(u) \end{pmatrix} \tag{20}$$

where $a(u), b(u), c(u), d(u), e(u)$ are as in (11) and u is the spectral parameter. The elements of the r-matrix $r(u)$ in (20) are equal to those of the original R-matrix $R(u)$ of (11) when its indices just take values 2, 3, 4. One can show that the r-matrix (20) is a solution of the graded Yang–Baxter equation

$$r_{12}(u-v)r_{13}(u)r_{23}(v) = r_{23}(v)r_{13}(u)r_{12}(u-v) \tag{21}$$

where $r_{12}(u) = r(u)\otimes I$, $r_{23}(u) = I\otimes r(u)$ etc.

Let the elements of $\mathcal{T}(u)$ in (17) act on the pseudovacuum state $|0\rangle = \otimes_{k=1}^{L}|0\rangle_k$, $|0\rangle_k = (1,0,0,0)^T$. Then we have

$$\begin{aligned} &T_{11}(u)|0\rangle = |0\rangle \qquad && T_{dd}(u)|0\rangle = \left(\frac{\sinh\frac{\gamma u}{2}}{\sinh\frac{\gamma(u-2)}{2}}\right)^L|0\rangle \\ &T_{1d}(u)|0\rangle \neq 0 && T_{db}(u)|0\rangle = 0 \qquad T_{d1}(u)|0\rangle = 0 \\ &\tilde{T}_{11}(u)|0\rangle = |0\rangle && \tilde{T}_{dd}(u)|0\rangle = \left(\frac{\sinh\frac{\gamma u}{2}}{\sinh\frac{\gamma(u-2)}{2}}\right)^L|0\rangle \\ &\tilde{T}_{1d}(u)|0\rangle \neq 0 && \tilde{T}_{db}(u)|0\rangle = 0 \qquad \tilde{T}_{d1}(u)|0\rangle = 0 \end{aligned} \tag{22}$$

where $d \neq b$, and $d, b = 2, 3, 4$, and

$$\begin{aligned} &T_{21}(u)\tilde{T}_{12}(u)|0\rangle = -\frac{e^{\gamma u}\sinh\gamma}{\sinh\gamma(u-1)}[\tilde{T}_{11}(u)T_{11}(u) - T_{22}(u)\tilde{T}_{22}(u)]|0\rangle \\ &T_{\alpha 1}(u)\tilde{T}_{1\alpha}(u)|0\rangle = 0 \qquad \alpha = 3, 4 \\ &T_{\alpha 1}(u)\tilde{T}_{1\beta}(u)|0\rangle = 0 \qquad \alpha \neq \beta. \end{aligned} \tag{23}$$

Then

$$\begin{aligned} &\mathcal{A}(u)|0\rangle = U_1(u)|0\rangle \\ &\mathcal{C}_a(u)|0\rangle = 0 \qquad \mathcal{B}_a(u)|0\rangle \neq 0 \\ &\check{\mathcal{D}}_{aa}(u)|0\rangle = W_a(u)\left(\frac{\sinh\frac{\gamma u}{2}}{\sinh\frac{\gamma(u-2)}{2}}\right)^{2L}|0\rangle. \end{aligned} \tag{24}$$

Here we have defined

$$\begin{aligned} &U_1(u) = \begin{cases} \dfrac{\sinh\frac{\gamma(u+\xi_-^I)}{2}}{\sinh\frac{\gamma\xi_-^I}{2}} & \text{for} \quad \text{Case I} \\ \dfrac{\sinh\frac{\gamma(u+\xi_-^{II})}{2}}{\sinh\frac{\gamma\xi_-^{II}}{2}} & \text{for} \quad \text{Case II} \\ \dfrac{\sinh\frac{\gamma(u+\xi_-^{III})}{2}}{\sinh\frac{\gamma\xi_-^{III}}{2}} & \text{for} \quad \text{Case III} \end{cases} \\ &W_a(u) = k_{a+1}(u) - \frac{\sinh\frac{\gamma\eta}{2}\, e^{\gamma u}}{\sinh\frac{\gamma(2u+\eta)}{2}}k_1(u) \qquad a = 1, 2, 3. \end{aligned} \tag{25}$$

Substituting the exact forms of the type I, type II and type III boundary K-matrices $K_-^I(u)$, $K_-^{II}(u)$ and $K_-^{III}(u)$ (15) into the relation (25), we have

$$W_1(u) = \begin{cases} -\dfrac{\sinh\gamma u\sinh\frac{\gamma(u-\xi_-^I-2)}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\xi_-^I}{2}}\, e^{\gamma u} & \text{for} \quad \text{Case I} \\ -\dfrac{\sinh\gamma u\sinh\frac{\gamma(u-\xi_-^{II}-2)}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\xi_-^{II}}{2}}\, e^{\gamma u} & \text{for} \quad \text{Case II} \\ \dfrac{\sinh\gamma u\sinh\frac{\gamma(u+\xi_-^{III})}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\xi_-^{III}}{2}}\, e^{\gamma} & \text{for} \quad \text{Case III} \end{cases}$$

$$W_2(u) = \begin{cases} \dfrac{\sinh\gamma u\sinh\frac{\gamma(u+\xi_-^I)}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\xi_-^I}{2}}\, e^{\gamma(2u-1)} & \text{for} \quad \text{Case I} \\ -\dfrac{\sinh\gamma u\sinh\frac{\gamma(u-\xi_-^{II}-2)}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\xi_-^{II}}{2}}\, e^{\gamma u} & \text{for} \quad \text{Case II} \\ -\dfrac{\sinh\gamma u\sinh\frac{\gamma(u-\xi_-^{III}-2)}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\xi_-^{III}}{2}}\, e^{\gamma u} & \text{for} \quad \text{Case III} \end{cases}$$

$$W_3(u) = \begin{cases} \dfrac{\sinh\gamma u \sinh\frac{\gamma(u+\xi_-^I)}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\xi_-^I}{2}} e^{\gamma(2u-1)} & \text{for} \quad \text{Case I} \\ \dfrac{\sinh\gamma u \sinh\frac{\gamma(u+\xi_-^{II})}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\xi_-^{II}}{2}} e^{\gamma(2u-1)} & \text{for} \quad \text{Case II} \\ -\dfrac{\sinh\gamma u \sinh\frac{\gamma(u-\xi_-^{III}-2)}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\xi_-^{III}}{2}} e^{\gamma u} & \text{for} \quad \text{Case III.} \end{cases}$$

The boundary transfer matrix with FFB grading is written as

$$\begin{aligned} \tau(u) &= k_1^+(u)\mathcal{A}(u) + k_2^+(u)\mathcal{D}_{11}(u) - k_3^+(u)\mathcal{D}_{22}(u) - k_4^+(u)\mathcal{D}_{33}(u) \\ &= U_1^+(u)\mathcal{A}(u) + k_2^+(u)\tilde{\mathcal{D}}_{11}(u) - k_3^+(u)\tilde{\mathcal{D}}_{22}(u) - k_4^+(u)\tilde{\mathcal{D}}_{33}(u) \end{aligned} \tag{26}$$

where U_1^+ is defined by

$$U_1^+(u) \equiv k_1^+(u) + \frac{e^{\gamma u}\sinh\frac{\gamma\eta}{2}}{\sinh\frac{\gamma(2u+\eta)}{2}}\left(k_2^+(u) - k_3^+(u) - k_4^+(u)\right).$$

For the type I, type II and type III boundary K-matrices $K_+^I(u)$, $K_+^{II}(u)$ and $K_+^{III}(u)$ (16), we have

$$U_1^+(u) = \begin{cases} \dfrac{\sinh\gamma u \sinh\frac{\gamma\left(u+\xi_+^I\right)}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\left(\xi_+^I-2\right)}{2}} e^{-2\gamma} & \text{for} \quad \text{Case I} \\ \dfrac{\sinh\gamma u \sinh\frac{\gamma\left(u+\xi_+^{II}\right)}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\xi_+^{II}}{2}} e^{-\gamma} & \text{for} \quad \text{Case II} \\ \dfrac{\sinh\gamma u \sinh\frac{\gamma\left(u+\xi_+^{III}-4\right)}{2}}{\sinh\gamma(u-1)\sinh\frac{\gamma\xi_+^{III}}{2}} e^{-\gamma} & \text{for} \quad \text{Case III.} \end{cases}$$

Using the graded algebraic Bethe ansatz method, we act with the boundary transfer matrix (26) on the vector

$$\mathcal{C}_{d_1}(u_1)\mathcal{C}_{d_2}(u_2)\cdots\mathcal{C}_{d_n}(u_n)|0\rangle F^{d_1\cdots d_n}$$

where the indices d_j run over the values $1, 2, 3$, and $F^{d_1\cdots d_n}$ is a function of the spectral parameters u_j. We have

$$\begin{aligned} &\tau(u)\mathcal{C}_{d_1}(u_1)\mathcal{C}_{d_2}(u_2)\cdots\mathcal{C}_{d_n}(u_n)|0\rangle F^{d_1\cdots d_n} \\ &\quad = U_1^+(u)U_1(u)\prod_{j=1}^{n}\frac{\sinh\frac{\gamma(u+u_j)}{2}\sinh\frac{\gamma(u-u_j-\eta)}{2}}{\sinh\frac{\gamma(u+u_j+\eta)}{2}\sinh\frac{\gamma(u-u_j)}{2}}\mathcal{C}_{d_1}(u_1)\cdots\mathcal{C}_{d_n}(u_n)|0\rangle F^{d_1\cdots d_n} \\ &\qquad + \left(\frac{\sinh\frac{\gamma u}{2}}{\sinh\frac{\gamma(u+\eta)}{2}}\right)^{2L}\prod_{j=1}^{n}\frac{\sinh\frac{\gamma(u+u_j+2\eta)}{2}\sinh\frac{\gamma(u-u_j+\eta)}{2}}{\sinh\frac{\gamma(u+u_j+\eta)}{2}\sinh\frac{\gamma(u-u_j)}{2}}\mathcal{C}_{c_1}(u_1)\cdots\mathcal{C}_{c_n} \\ &\qquad \times (u_n)|0\rangle\tau^{(1)}(u)_{d_1\cdots d_n}^{c_1\cdots c_n}F^{d_1\cdots d_n} + \text{u.t.} \end{aligned} \tag{27}$$

where u.t. means unwanted terms. The eigenvalue $\Lambda(u)$ of the boundary transfer matrix $\tau(u)$ is finally obtained as

$$\Lambda(u) = U_1^+(u)U_1(u)\prod_{j=1}^{n}\frac{\sinh\frac{\gamma(u+u_j)}{2}\sinh\frac{\gamma(u-u_j-\eta)}{2}}{\sinh\frac{\gamma(u+u_j+\eta)}{2}\sinh\frac{\gamma(u-u_j)}{2}}$$
$$+\left(\frac{\sinh\frac{\gamma u}{2}}{\sinh\frac{\gamma(u+\eta)}{2}}\right)^{2L}\prod_{j=1}^{n}\frac{\sinh\frac{\gamma(u+u_j+2\eta)}{2}\sinh\frac{\gamma(u-u_j+\eta)}{2}}{\sinh\frac{\gamma(u+u_j+\eta)}{2}\sinh\frac{\gamma(u-u_j)}{2}}\Lambda^{(1)}(u,\{u_j\}) \tag{28}$$

provided the parameters $\{u_j\}$ satisfy

$$U_1^+(u_j)U_1(u_j)\frac{\sinh\gamma u_j}{\sinh\gamma(u_j-2)}\left(\frac{\sinh\frac{\gamma(u_j-2)}{2}}{\sinh\frac{\gamma u_j}{2}}\right)^{2L}$$
$$=\prod_{\substack{i=1\\ i\neq j}}^{n}\frac{\sinh\frac{\gamma(u_j+u_i-4)}{2}\sinh\frac{\gamma(u_j-u_i-2)}{2}}{\sinh\frac{\gamma(u_j+u_i)}{2}\sinh\frac{\gamma(u_j-u_i+2)}{2}}\Lambda^{(1)}(u_j). \tag{29}$$

We denote $\tilde{u} = u-1$, $\tilde{\xi}_-^{I,II} = \xi_-^{I,II}+1$, $\tilde{\xi}_-^{III} = -\xi_-^{III}-1$, $\tilde{\xi}_+^{I} = -\xi_+^{I}+1$, $\tilde{\xi}_+^{II} = -\xi_+^{II}-1$, $\tilde{\xi}_+^{III} = \xi_+^{III}-1$, and

$$U_1^+(\tilde{u}_j)U_1(\tilde{u}_j)\frac{\sinh\gamma(\tilde{u}_j+1)}{\sinh\gamma(\tilde{u}_j-1)}\left(\frac{\sinh\frac{\gamma(\tilde{u}_j-1)}{2}}{\sinh\frac{\gamma(\tilde{u}_j+1)}{2}}\right)^{2L}$$
$$=\prod_{\substack{i=1\\ i\neq j}}^{n}\frac{\sinh\frac{\gamma(\tilde{u}_j+\tilde{u}_i-2)}{2}\sinh\frac{\gamma(\tilde{u}_j-\tilde{u}_i-2)}{2}}{\sinh\frac{\gamma(\tilde{u}_j+\tilde{u}_i+2)}{2}\sinh\frac{\gamma(\tilde{u}_j-\tilde{u}_i+2)}{2}}\Lambda^{(1)}(\tilde{u}_j) \tag{30}$$

where

$$U_1(\tilde{u}) = \begin{cases} \dfrac{\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{I})}{2}}{\sinh\frac{\gamma(\tilde{\xi}_-^{I}-1)}{2}} & \text{for}\quad \text{Case I}\\[2ex] \dfrac{\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{II})}{2}}{\sinh\frac{\gamma(\tilde{\xi}_-^{II}-1)}{2}} & \text{for}\quad \text{Case II}\\[2ex] \dfrac{\sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_-^{III})}{2}}{\sinh\frac{\gamma(\tilde{\xi}_-^{III}+1)}{2}} & \text{for}\quad \text{Case III}\end{cases}$$

and

$$U_1^+(\tilde{u}) = \begin{cases} -\dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma\left(\tilde{u}-\tilde{\xi}_+^{I}+2\right)}{2}}{\sinh\gamma\tilde{u}\sinh\frac{\gamma\left(\tilde{\xi}_+^{I}+1\right)}{2}}\mathrm{e}^{-2\gamma} & \text{for}\quad \text{Case I}\\[2ex] \dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma\left(-\tilde{u}+\tilde{\xi}_+^{II}\right)}{2}}{\sinh\gamma\tilde{u}\sinh\frac{\gamma\left(\tilde{\xi}_+^{II}+1\right)}{2}}\mathrm{e}^{-\gamma} & \text{for}\quad \text{Case II}\\[2ex] \dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma\left(\tilde{u}+\tilde{\xi}_+^{III}-2\right)}{2}}{\sinh\gamma\tilde{u}\sinh\frac{\gamma\left(\tilde{\xi}_+^{III}+1\right)}{2}}\mathrm{e}^{-\gamma} & \text{for}\quad \text{Case III.}\end{cases}$$

The nested boundary transfer matrix $\tau^{(1)}(u)$ can still be interpreted as a boundary transfer matrix with open boundary conditions corresponding to the anisotropic case,

$$\tau^{(1)}(\tilde{u}) = \mathrm{str}\left(K_+^{(1)}(\tilde{u})T^{(1)}(\tilde{u},\{\tilde{u}_j\})K_-^{(1)}(\tilde{u})T^{(1)^{-1}}(-\tilde{u},\{\tilde{u}_j\})\right) \tag{31}$$

with the bosonic, fermionic and fermionic (BFF) grading, which means $[|1\rangle^{(1)}] = 0$, $[|2\rangle^{(1)}] = [|3\rangle^{(1)}] = 1$. According to the definition, we have the nested boundary K-matrices $K_\pm^{(1)}(\tilde{u})$:

$$\begin{aligned}
K_-^{(1)}(\tilde{u}) &\equiv \begin{pmatrix} k_1^{(1)}(\tilde{u}) & & \\ & k_2^{(1)}(\tilde{u}) & \\ & & k_3^{(1)}(\tilde{u}) \end{pmatrix} = \begin{pmatrix} W_1(\tilde{u}+1) & & \\ & W_2(\tilde{u}+1) & \\ & & W_3(\tilde{u}+1) \end{pmatrix} \\
&= \begin{cases}
\dfrac{\sinh\gamma(\tilde{u}+1)}{\sinh\gamma\tilde{u}\sinh\frac{\gamma(\tilde{\xi}_-^{I}-1)}{2}}\, \mathrm{e}^{\gamma(\tilde{u}+1)}\,\mathrm{diag}\left(\sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_-^{I})}{2}, \sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{I})}{2}\mathrm{e}^{\gamma\tilde{u}},\right. \\
\qquad\qquad \left.\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{I})}{2}\mathrm{e}^{\gamma\tilde{u}}\right) \qquad \text{for}\quad \text{Case I} \\
\dfrac{\sinh\gamma(\tilde{u}+1)}{\sinh\gamma\tilde{u}\sinh\frac{\gamma(\tilde{\xi}_-^{II}-1)}{2}}\, \mathrm{e}^{\gamma(\tilde{u}+1)}\,\mathrm{diag}\left(\sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_-^{II})}{2}, \sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_-^{I})}{2},\right. \\
\qquad\qquad \left.\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{II})}{2}\mathrm{e}^{\gamma\tilde{u}}\right) \qquad \text{for}\quad \text{Case II} \\
\dfrac{\sinh\gamma(\tilde{u}+1)}{\sinh\gamma\tilde{u}\sinh\frac{\gamma(\tilde{\xi}_-^{III}+1)}{2}}\, \mathrm{e}^{\gamma}\,\mathrm{diag}\left(\sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_-^{III})}{2}, \sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{III})}{2}\mathrm{e}^{\gamma\tilde{u}},\right. \\
\qquad\qquad \left.\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{III})}{2}\mathrm{e}^{\gamma\tilde{u}}\right) \qquad \text{for}\quad \text{Case III}
\end{cases}
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
K_+^{(1)}(\tilde{u}) &\equiv \begin{pmatrix} k_1^{(1)^+}(\tilde{u}) & & \\ & k_2^{(1)^+}(\tilde{u}) & \\ & & k_3^{(1)^+}(\tilde{u}) \end{pmatrix} = \begin{pmatrix} k_2^+(\tilde{u}+1) & & \\ & k_3^+(\tilde{u}+1) & \\ & & k_4^+(\tilde{u}+1) \end{pmatrix} \\
&= \begin{cases}
\dfrac{\mathrm{e}^{-\gamma(\tilde{u}+3)}}{\sinh\frac{\gamma(\tilde{\xi}_+^{I}+1)}{2}}\,\mathrm{diag}\left(\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_+^{I}+2)}{2}, \sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_+^{I})}{2}\mathrm{e}^{-\gamma(\tilde{u}+1)},\right. \\
\qquad\qquad \left.\sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_+^{I})}{2}\mathrm{e}^{-\gamma(\tilde{u}-1)}\right) \qquad \text{for}\quad \text{Case I} \\
\dfrac{\mathrm{e}^{-\gamma(\tilde{u}+3)}}{\sinh\frac{\gamma(\tilde{\xi}_+^{II}+1)}{2}}\,\mathrm{diag}\left(\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_+^{II}+2)}{2}, \sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_+^{II}+2)}{2},\right. \\
\qquad\qquad \left.\sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_+^{II})}{2}\mathrm{e}^{-\gamma(\tilde{u}-1)}\right) \qquad \text{for}\quad \text{Case II} \\
\dfrac{\mathrm{e}^{-2\gamma}}{\sinh\frac{\gamma(\tilde{\xi}_+^{III}+1)}{2}}\,\mathrm{diag}\left(\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_+^{III}+2)}{2}, \sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_+^{III})}{2}\mathrm{e}^{-\gamma(\tilde{u}+1)},\right. \\
\qquad\qquad \left.\sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_+^{III})}{2}\mathrm{e}^{-\gamma(\tilde{u}-1)}\right) \qquad \text{for}\quad \text{Case III.}
\end{cases}
\end{aligned} \tag{33}$$

Corresponding to $K_-^{(1)}(\tilde{u})$, the nested monodromy matrices $T^{(1)}(\tilde{u},\{\tilde{u}_j\})$ and $T^{(1)^{-1}}(-\tilde{u},\{\tilde{u}_j\})$ are defined respectively as

$$T^{(1)}(\tilde{u},\{\tilde{u}_j\}) = L_1^{(1)}(\tilde{u}+\tilde{u}_1)L_2^{(1)}(\tilde{u}+\tilde{u}_2)\cdots L_n^{(1)}(\tilde{u}+\tilde{u}_n)$$

where the local L-operator is obtained from the r-matrix and takes the form

$$L_k^{(1)}(\tilde{u}) = \begin{pmatrix} b(\tilde{u}) - (b(\tilde{u}) - a(\tilde{u}))e_{11}^k & c(\tilde{u})e_{21}^k & c(\tilde{u})e_{31}^k \\ d(\tilde{u})e_{12}^k & b(\tilde{u}) - (b(\tilde{u}) - e(\tilde{u}))e_{22}^k & -c(\tilde{u})e_{32}^k \\ d(\tilde{u})e_{13}^k & -d(\tilde{u})e_{23}^k & b(\tilde{u}) - (b(\tilde{u}) - e(\tilde{u}))e_{33}^k \end{pmatrix}$$

and

$$T^{(1)^{-1}}(-\tilde{u}, \{\tilde{u}_j\}) = L_n^{(1)^{-1}}(-\tilde{u} + \tilde{u}_n) \cdots L_2^{(1)^{-1}}(-\tilde{u} + \tilde{u}_2) L_1^{(1)^{-1}}(-\tilde{u} + \tilde{u}_1).$$

Here we have used the unitarity relation of the r-matrix,

$$r_{12}(u) r_{21}(-u) = \sinh \frac{\gamma(-u + \eta)}{2} \sinh \frac{\gamma(u + \eta)}{2}.$$

Therefore, we have the graded Yang–Baxter relation for the nested monodromy matrix,

$$r_{12}(u - v) \overset{1}{T}{}^{(1)}(u, \{u_j\}) \overset{2}{T}{}^{(1)}(v, \{u_j\}) = \overset{2}{T}{}^{(1)}(v, \{u_j\}) \overset{1}{T}{}^{(1)}(u, \{u_j\}) r_{12}(u - v). \tag{34}$$

Here $\overset{1}{T}{}^{(1)}(u, \{u_j\}) = T_{13}^{(1)}(u, \{u_j\})$, $\overset{2}{T}{}^{(1)}(u, \{u_j\}) = T_{23}^{(1)}(u, \{u_j\})$. In order to prove that the $\tau^{(1)}(u)$ in (31) is indeed the boundary transfer matrix with open boundary conditions, we need to prove that $K_{\pm}^{(1)}$ satisfy the nested graded reflection equation and the dual nested graded reflection equation. By a direct calculation, we can prove that the boundary K-matrices $K_{-}^{(1)^{I}}$, $K_{-}^{(1)^{II}}$ and $K_{-}^{(1)^{III}}$ (32) satisfy the nested graded reflection equation:

$$r_{12}(u - v) \overset{1}{K}{}_{-}^{(1)}(u) r_{21}(u + v) \overset{2}{K}{}_{-}^{(1)}(v) = \overset{2}{K}{}_{-}^{(1)}(v) r_{12}(u + v) \overset{1}{K}{}_{-}^{(1)}(u) r_{21}(u - v). \tag{35}$$

We note that the r-matrix also satisfies the crossing-unitarity relation,

$$r_{12}^{st_1}(-u - 2) \overset{1}{M}{}^{(1)} r_{21}^{st_1}(u) \overset{1}{M}{}^{(1)^{-1}} = \sinh \frac{\gamma u}{2} \sinh \frac{\gamma(-u - \eta)}{2}$$

where $M^{(1)}$ is the crossing matrix: $M^{(1)} = \mathrm{diag}(1, 1, \mathrm{e}^{-\eta\gamma})$. We can also prove that $K_{+}^{(1)^{I}}$, $K_{+}^{(1)^{II}}$ and $K_{+}^{(1)^{III}}$ (33) satisfy the dual nested graded reflection equation:

$$\begin{aligned} & r_{12}(-u + v) \overset{1}{K}{}_{+}^{(1)}(u) \overset{1}{M}{}^{(1)^{-1}} r_{21}(-u - v + \eta) \overset{1}{M}{}^{(1)} \overset{2}{K}{}_{+}^{(1)}(v) \\ & \quad = \overset{1}{M}{}^{(1)} \overset{2}{K}{}_{+}^{(1)}(v) r_{12}(-u - v + \eta) \overset{1}{M}{}^{(1)^{-1}} \overset{1}{K}{}_{+}^{(1)}(u) r_{21}(-u + v). \end{aligned} \tag{36}$$

One finds that there is an isomorphism between (35) and (36) for the case $g = \frac{\eta}{2}$,

$$K_{+}^{(1)}(u) = M^{(1)} K_{-}^{(1)}\left(-u + \frac{\eta}{2}\right).$$

Thus the nested boundary transfer matrices $\tau^{(1)}(u)$ are shown to constitute a commuting family. We can still use the graded algebraic Bethe ansatz method to find the eigenvalues and eigenvectors of the $\tau^{(1)}(u)$.

3.3. The nested algebraic Bethe ansatz method

Now, let us use again the graded algebraic Bethe ansatz method to obtain the eigenvalue $\Lambda^{(1)}(u)$ of the nested boundary transfer matrix $\tau^{(1)}(u)$. We write the nested doubled monodromy matrix as

$$\mathcal{T}^{(1)}(u,\{u_i\}) = T^{(1)}(u,\{u_i\})K_-^{(1)}(u)T^{(1)^{-1}}(-u,\{u_i\})$$
$$= \begin{pmatrix} \mathcal{A}^{(1)}(u) & \mathcal{B}_1^{(1)}(u) & \mathcal{B}_2^{(1)}(u) \\ \mathcal{C}_1^{(1)}(u) & \mathcal{D}_{11}^{(1)}(u) & \mathcal{D}_{12}^{(1)}(u) \\ \mathcal{C}_2^{(1)}(u) & \mathcal{D}_{21}^{(1)}(u) & \mathcal{D}_{22}^{(1)}(u) \end{pmatrix}. \tag{37}$$

Since we know that $K_-^{(1)}(u)$ satisfies (35), we can show that the $\mathcal{T}^{(1)}(u,\{v_i\})$ in (55) satisfies the equation

$$r_{12}(u-v)\overset{1}{\mathcal{T}^{(1)}}(u)r_{21}(u+v)\overset{2}{\mathcal{T}^{(1)}}(v) = \overset{2}{\mathcal{T}^{(1)}}(v)r_{12}(u+v)\overset{1}{\mathcal{T}^{(1)}}(u)r_{21}(u-v). \tag{38}$$

For convenience, we introduce again a transformation

$$\mathcal{D}_{ab}^{(1)}(u) = \check{\mathcal{D}}_{ab}^{(1)}(u) + \delta_{ab}\frac{\sinh\frac{\gamma\eta}{2}\,\mathrm{e}^{\gamma u}}{\sinh\frac{\gamma(2u+\eta)}{2}}\mathcal{A}^{(1)}(u).$$

In this case, we can find the following commutation relations:

$$\begin{aligned}\check{\mathcal{D}}_{bd}^{(1)}(u)\mathcal{B}_c^{(1)}(v) &= \frac{\sinh\frac{\gamma(u-v+2)}{2}\sinh\frac{\gamma(u+v)}{2}}{\sinh\frac{\gamma(u-v)}{2}\sinh\frac{\gamma(u+v-2)}{2}}r^{(1)}(u+v-2)_{gh}^{eb}r^{(1)}(u-v)_{cd}^{ih}\mathcal{B}_e^{(1)}(v)\check{\mathcal{D}}_{gi}^{(1)}(u)\\ &\quad+\frac{\mathrm{e}^{\frac{\gamma(u+v-2)}{2}}\sinh\gamma\sinh\gamma u\sinh\gamma v}{\sinh\frac{\gamma(u+v-2)}{2}\sinh\gamma(u-1)\sinh\gamma(v-1)}r^{(1)}(2u-2)_{cd}^{gb}\mathcal{B}_g^{(1)}(u)\mathcal{A}^{(1)}(v)\\ &\quad-\frac{\mathrm{e}^{\frac{\gamma(u-v)}{2}}\sinh\gamma\sinh\gamma u}{\sinh\frac{\gamma(u-v)}{2}\sinh\gamma(u-1)}r^{(1)}(2u-2)_{id}^{gb}\mathcal{B}_g^{(1)}(u)\check{\mathcal{D}}_{ic}^{(1)}(v)\end{aligned} \tag{39}$$

$$\begin{aligned}\mathcal{A}^{(1)}(u)\mathcal{B}_b^{(1)}(v) &= \frac{\sinh\frac{\gamma(u-v+2)}{2}\sinh\frac{\gamma(u+v)}{2}}{\sinh\frac{\gamma(u-v)}{2}\sinh\frac{\gamma(u+v-2)}{2}}\mathcal{B}_b^{(1)}(v)\mathcal{A}^{(1)}(u)\\ &\quad+\frac{\mathrm{e}^{-\frac{\gamma(u-v)}{2}}\sinh\gamma\sinh\gamma v}{\sinh\gamma(v-1)\sinh\frac{\gamma(u-v)}{2}}\mathcal{B}_b^{(1)}(u)\mathcal{A}^{(1)}(v)\\ &\quad+\frac{\mathrm{e}^{-\frac{\gamma(u+v)}{2}}\sinh\gamma}{\sinh\frac{\gamma(u+v-2)}{2}}\mathcal{B}_d^{(1)}(u)\check{\mathcal{D}}_{db}^{(1)}(v).\end{aligned} \tag{40}$$

Now the indices take values 1, 2, and the $r^{(1)}$-matrix is defined as

$$r^{(1)}(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sinh\frac{\gamma u}{2}}{\sinh\frac{\gamma(u-\eta)}{2}} & -\frac{\sinh\frac{\gamma u}{2}\,\mathrm{e}^{-\frac{\gamma u}{2}}}{\sinh\frac{\gamma(u-\eta)}{2}} & 0 \\ 0 & -\frac{\sinh\frac{\gamma u}{2}\,\mathrm{e}^{\frac{\gamma u}{2}}}{\sinh\frac{\gamma(u-\eta)}{2}} & \frac{\sinh\frac{\gamma u}{2}}{\sinh\frac{\gamma(u-\eta)}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{41}$$

In this case, we can show that this r-matrix $r^{(1)}(u)$ is a solution of the graded Yang–Baxter equation

$$r_{12}^{(1)}(u-v)r_{13}^{(1)}(u)r_{23}^{(1)}(v) = r_{23}^{(1)}(v)r_{13}^{(1)}(u)r_{12}^{(1)}(u-v) \tag{42}$$

where $r_{12}^{(1)}(u) = r^{(1)}(u)\otimes I$, $r_{23}^{(1)}(u) = I\otimes r^{(1)}(u)$ etc.

Letting the elements of $\mathcal{T}^{(1)}(u)$ act on the pseudovacuum state $|0\rangle^{(1)} = \otimes_{k=1}^{m_1}|0\rangle_k^{(1)}$, $|0\rangle_k^{(1)} = (1,0,0)^T$, we have

$$\begin{aligned}
&\mathcal{A}(\tilde{u})|0\rangle^{(1)} = U_1^{(1)}(\tilde{u})|0\rangle^{(1)} \\
&\mathcal{C}_a^{(1)}(\tilde{u})|0\rangle^{(1)} = 0 \qquad \mathcal{B}_a^{(1)}(\tilde{u})|0\rangle^{(1)} \neq 0 \\
&\check{\mathcal{D}}_{dd}^{(1)}(\tilde{u})|0\rangle^{(1)} = W_d^{(1)}(\tilde{u})\prod_{I=1}^{n}\frac{\sinh\frac{\gamma(\tilde{u}+\tilde{u}_i)}{2}}{\sinh\frac{\gamma(\tilde{u}+\tilde{u}_i-2)}{2}}\frac{\sinh\frac{\gamma(\tilde{u}-\tilde{u}_i)}{2}}{\sinh\frac{\gamma(\tilde{u}-\tilde{u}_i-2)}{2}}|0\rangle^{(1)} \\
&\check{\mathcal{D}}_{bd}^{(1)}(\tilde{u})|0\rangle^{(1)} = 0.
\end{aligned} \tag{43}$$

Here $U_1^{(1)}$ and $W_d^{(1)}$ take the form

$$U_1^{(1)}(\tilde{u}) = k_1^{(1)}(\tilde{u}) \qquad W_d^{(1)}(\tilde{u}) = k_{d+1}^{(1)}(\tilde{u}) + \frac{\sinh\frac{\gamma\eta}{2}\,\mathrm{e}^{\gamma\tilde{u}}}{\sinh\frac{\gamma(2\tilde{u}+\eta)}{2}}k_1^{(1)}(\tilde{u}) \qquad d = 1,2 \tag{44}$$

so that

$$U_1^{(1)}(\tilde{u}) = k_1^{(1)}(\tilde{u}) = \begin{cases}
\dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{I})}{2}}{\sinh\gamma\tilde{u}\sinh\frac{\gamma(\tilde{\xi}_-^{I}-1)}{2}}\,\mathrm{e}^{\gamma(\tilde{u}+1)} & \text{for Case I} \\
\dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_-^{II})}{2}}{\sinh\gamma\tilde{u}\sinh\frac{\gamma(\tilde{\xi}_-^{II}-1)}{2}}\,\mathrm{e}^{\gamma(\tilde{u}+1)} & \text{for Case II} \\
\dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_-^{III})}{2}}{\sinh\gamma\tilde{u}\sinh\frac{\gamma(\tilde{\xi}_-^{III}-1)}{2}}\,\mathrm{e}^{\gamma} & \text{for Case III}
\end{cases}$$

$$W_1^{(1)}(\tilde{u}) = \begin{cases}
\dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{I}-2)}{2}}{\sinh\gamma(\tilde{u}-1)\sinh\frac{\gamma(\tilde{\xi}_-^{I}-1)}{2}}\,\mathrm{e}^{\gamma(2\tilde{u}+1)} & \text{for Case I} \\
\dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma(-\tilde{u}+\tilde{\xi}_-^{II})}{2}}{\sinh\gamma(\tilde{u}-1)\sinh\frac{\gamma(\tilde{\xi}_-^{II}-1)}{2}}\,\mathrm{e}^{\gamma(\tilde{u}+2)} & \text{for Case II} \\
\dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{III}-2)}{2}}{\sinh\gamma(\tilde{u}-1)\sinh\frac{\gamma(\tilde{\xi}_-^{III}+1)}{2}}\,\mathrm{e}^{\gamma(\tilde{u}+1)} & \text{for Case III}
\end{cases}$$

$$W_2^{(1)}(\tilde{u}) = \begin{cases}
\dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{I}-2)}{2}}{\sinh\gamma(\tilde{u}-1)\sinh\frac{\gamma(\tilde{\xi}_-^{I}-1)}{2}}\,\mathrm{e}^{\gamma(2\tilde{u}+1)} & \text{for Case I} \\
\dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{II}-2)}{2}}{\sinh\gamma(\tilde{u}-1)\sinh\frac{\gamma(\tilde{\xi}_-^{II}-1)}{2}}\,\mathrm{e}^{\gamma(2\tilde{u}+1)} & \text{for Case II} \\
\dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma(\tilde{u}+\tilde{\xi}_-^{III}-2)}{2}}{\sinh\gamma(\tilde{u}-1)\sinh\frac{\gamma(\tilde{\xi}_-^{III}+1)}{2}}\,\mathrm{e}^{\gamma(\tilde{u}+1)} & \text{for Case III.}
\end{cases}$$

The nested boundary transfer matrix takes the form

$$\begin{aligned}\tau^{(1)}(\tilde{u}) &= \mathrm{str}(K^{(1)}(\tilde{u})\mathcal{T}^{(1)}(\tilde{u})) \\ &= k_1^{(1)^+}(\tilde{u})\mathcal{A}^{(1)}(\tilde{u}) - k_2^{(1)^+}(\tilde{u})\mathcal{D}_{11}^{(1)}(\tilde{u}) - k_3^{(1)^+}(\tilde{u})\mathcal{D}_{11}^{(1)}(\tilde{u}) \\ &= U_1^{(1)^+}(\tilde{u})\mathcal{A}^{(1)}(\tilde{u}) - k_2^{(1)^+}(\tilde{u})\check{\mathcal{D}}_{11}^{(1)}(\tilde{u}) - k_3^{(1)^+}(\tilde{u})\check{\mathcal{D}}_{22}^{(1)}(\tilde{u})\end{aligned} \tag{45}$$

where we denote

$$U_1^{(1)^+}(\tilde{u}) = k_1^{(1)^+}(\tilde{u}) - \frac{\sinh\frac{\gamma\eta}{2}\,\mathrm{e}^{\gamma\tilde{u}}}{\sinh\frac{\gamma(2\tilde{u}+\eta)}{2}}\left(k_2^{(1)^+}(\tilde{u}) + k_3^{(1)^+}(\tilde{u})\right). \tag{46}$$

Thus

$$U_1^{(1)^+}(\tilde{u}) = \begin{cases} \dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma\left(\tilde{u}+\tilde{\xi}_+^{I}-2\right)}{2}}{\sinh\gamma(\tilde{u}-1)\sinh\frac{\gamma\left(\tilde{\xi}_+^{I}+1\right)}{2}}\,\mathrm{e}^{-\gamma(\tilde{u}+3)} & \text{for}\quad \text{Case I} \\ \dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma\left(\tilde{u}+\tilde{\xi}_+^{II}\right)}{2}}{\sinh\gamma(\tilde{u}-1)\sinh\frac{\gamma\left(\tilde{\xi}_+^{II}+1\right)}{2}}\,\mathrm{e}^{-\gamma(\tilde{u}+2)} & \text{for}\quad \text{Case II} \\ \dfrac{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma\left(\tilde{u}+\tilde{\xi}_+^{III}-2\right)}{2}}{\sinh\gamma(\tilde{u}-1)\sinh\frac{\gamma\left(\tilde{\xi}_+^{III}+1\right)}{2}}\,\mathrm{e}^{-2\gamma} & \text{for}\quad \text{Case III.} \end{cases}$$

Following the standard algebraic Bethe ansatz method, acting the nested boundary transfer matrix $\tau^{(1)}(\tilde{u})$ (45) on the states

$$\mathcal{C}\left(u_1^{(1)}\right)\mathcal{C}\left(u_2^{(1)}\right)\cdots\mathcal{C}\left(u_m^{(1)}\right)|0\rangle^{(1)}$$

we find the eigenvalue of $\tau^{(1)}(\tilde{u})$ as follows,

$$\begin{aligned}\Lambda^{(1)}(\tilde{u}) = U_1^{(1)^+}(\tilde{u})U_1^{(1)}(\tilde{u})\prod_{k=1}^{m_1}\frac{\sinh\frac{\gamma\left(\tilde{u}-\tilde{\tilde{u}}_k^{(1)}+2\right)}{2}\sinh\frac{\gamma\left(\tilde{u}+\tilde{\tilde{u}}_k^{(1)}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}-\tilde{\tilde{u}}_k^{(1)}\right)}{2}\sinh\frac{\gamma\left(\tilde{u}+\tilde{\tilde{u}}_k^{(1)}-2\right)}{2}} \\ + \prod_{j=1}^{n}\frac{\sinh\frac{\gamma(\tilde{u}+\tilde{u}_j)}{2}\sinh\frac{\gamma(\tilde{u}-\tilde{u}_j)}{2}}{\sinh\frac{\gamma(\tilde{u}+\tilde{u}_j-2)}{2}\sinh\frac{\gamma(\tilde{u}-\tilde{u}_j-2)}{2}}\prod_{k=1}^{m_1}\frac{\sinh\frac{\gamma\left(\tilde{u}-\tilde{\tilde{u}}_k^{(1)}+2\right)}{2}\sinh\frac{\gamma\left(\tilde{u}+\tilde{\tilde{u}}_k^{(1)}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}-\tilde{\tilde{u}}_k^{(1)}\right)}{2}\sinh\frac{\gamma\left(\tilde{u}+\tilde{\tilde{u}}_k^{(1)}-2\right)}{2}}\Lambda^{(2)}(\tilde{u})\end{aligned} \tag{47}$$

where $\left\{\tilde{\tilde{u}}_k^{(1)}\right\}$ satisfies the Bethe ansatz equation

$$\Lambda^{(2)}\left(\tilde{\tilde{u}}_k^{(1)}\right) = -U_1^{(1)^+}\left(\tilde{\tilde{u}}_k^{(1)}\right)U_1^{(1)}\left(\tilde{\tilde{u}}_k^{(1)}\right)\prod_{j=1}^{n}\frac{\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}+\tilde{u}_j-2\right)}{2}\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}-\tilde{u}_j-2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}+\tilde{u}_j\right)}{2}\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}-\tilde{u}_j\right)}{2}}. \tag{48}$$

The second nested boundary transfer matrix $\tau^{(2)}(u)$ can still be interpreted as a boundary transfer matrix with open boundary conditions corresponding to the anisotropic case,

$$\tau^{(2)}(\tilde{\tilde{u}}) = \mathrm{str}\left(K_+^{(2)}(\tilde{\tilde{u}})T^{(2)}(\tilde{\tilde{u}},\{\tilde{\tilde{u}}_j\})K_-^{(2)}(\tilde{\tilde{u}})T^{(2)^{-1}}(-\tilde{\tilde{u}},\{\tilde{\tilde{u}}_j\})\right) \tag{49}$$

with the fermionic and fermionic (FF) grading, which means $[|1\rangle^{(2)}] = [|2\rangle^{(2)}] = 1$, and we denote $\tilde{\tilde{u}} = \tilde{u} - 1$, $\tilde{\tilde{\xi}}_- = \tilde{\xi}_- + \frac{\eta}{2}$, $\tilde{\tilde{\xi}}_+ = \tilde{\xi}_+ - \frac{\eta}{2}$. According to the definition, we have the second nested boundary K-matrices $K_{\pm}^{(2)}(\tilde{\tilde{u}})$:

$$
\begin{aligned}
K_{-}^{(2)}(\tilde{\tilde{u}}) &\equiv \begin{pmatrix} k_1^{(2)}(\tilde{\tilde{u}}) & \\ & k_2^{(2)}(\tilde{\tilde{u}}) \end{pmatrix} = \begin{pmatrix} W_1^{(1)}(\tilde{\tilde{u}}+1) & \\ & W_2^{(1)}(\tilde{\tilde{u}}+1) \end{pmatrix} \\
&= \begin{cases} \dfrac{\sinh\gamma(\tilde{\tilde{u}}+2)\sinh\frac{\gamma(\tilde{\tilde{u}}+\tilde{\tilde{\xi}}_-^{I})}{2}}{\sinh\gamma\tilde{\tilde{u}}\sinh\frac{\gamma\tilde{\tilde{\xi}}_-^{I}}{2}}\,\mathrm{e}^{\gamma(2\tilde{\tilde{u}}+3)}\cdot I & \text{for} \quad \text{Case I} \\[2ex] \dfrac{\sinh\gamma(\tilde{\tilde{u}}+2)}{\sinh\gamma\tilde{\tilde{u}}\sinh\frac{\gamma\tilde{\tilde{\xi}}_-^{II}}{2}}\,\mathrm{e}^{\gamma(\tilde{\tilde{u}}+3)}\,\mathrm{diag}\left(\sinh\frac{\gamma(-\tilde{\tilde{u}}+\tilde{\tilde{\xi}}_-^{II})}{2}, \sinh\frac{\gamma(\tilde{\tilde{u}}+\tilde{\tilde{\xi}}_-^{II})}{2}\,\mathrm{e}^{\gamma\tilde{\tilde{u}}}\right) & \\ \qquad\qquad \text{for} \quad \text{Case II} & \\[2ex] \dfrac{\sinh\gamma(\tilde{\tilde{u}}+2)\sinh\frac{\gamma(\tilde{\tilde{u}}+\tilde{\tilde{\xi}}_-^{III})}{2}}{\sinh\gamma\tilde{\tilde{u}}\sinh\frac{\gamma(-\tilde{\tilde{u}}+\tilde{\tilde{\xi}}_-^{III})}{2}}\,\mathrm{e}^{\gamma(\tilde{\tilde{u}}+2)}\cdot I & \text{for} \quad \text{Case III} \end{cases}
\end{aligned}
\tag{50}
$$

and

$$
\begin{aligned}
K_{+}^{(2)}(\tilde{\tilde{u}}) &\equiv \begin{pmatrix} k_1^{(2)^+}(\tilde{\tilde{u}}) & \\ & k_2^{(2)^+}(\tilde{\tilde{u}}) \end{pmatrix} = \begin{pmatrix} k_2^{(1)^+}(\tilde{\tilde{u}}+1) & \\ & k_3^{(1)^+}(\tilde{\tilde{u}}+1) \end{pmatrix} \\
&= \begin{cases} \dfrac{\sinh\frac{\gamma\left(-\tilde{\tilde{u}}+\tilde{\tilde{\xi}}_+^{I}-2\right)}{2}}{\sinh\frac{\left(\gamma\tilde{\tilde{\xi}}_+^{I}\right)}{2}}\,\mathrm{e}^{-\gamma(2\tilde{\tilde{u}}+4)}\,\mathrm{diag}(\mathrm{e}^{-2\gamma},1) & \text{for} \quad \text{Case I} \\[2ex] \dfrac{\mathrm{e}^{-\gamma(\tilde{\tilde{u}}+4)}}{\sinh\frac{\gamma\tilde{\tilde{\xi}}_+^{II}}{2}}\,\mathrm{diag}\left(\sinh\frac{\gamma\left(\tilde{\tilde{u}}+\tilde{\tilde{\xi}}_+^{II}+2\right)}{2}, \sinh\frac{\gamma\left(-\tilde{\tilde{u}}+\tilde{\tilde{\xi}}_+^{II}-2\right)}{2}\,\mathrm{e}^{-\gamma\tilde{\tilde{u}}}\right) & \\ \qquad\qquad \text{for} \quad \text{Case II} & \\[2ex] \dfrac{\sinh\frac{\gamma\left(-\tilde{\tilde{u}}+\tilde{\tilde{\xi}}_+^{III}-2\right)}{2}}{\sinh\frac{\gamma\tilde{\tilde{\xi}}_+^{II}}{2}}\,\mathrm{e}^{-\gamma(\tilde{\tilde{u}}+2)}\,\mathrm{diag}(\mathrm{e}^{-2\gamma},1) & \text{for} \quad \text{Case III.} \end{cases}
\end{aligned}
\tag{51}
$$

Corresponding to $K_{-}^{(2)}(\tilde{\tilde{u}})$, the nested monodromy matrices $T^{(2)}(\tilde{\tilde{u}},\{\tilde{\tilde{u}}_j\})$ and $T^{(2)^{-1}}(-\tilde{\tilde{u}},\{\tilde{\tilde{u}}_j\})$ are defined respectively as

$$T^{(2)}(\tilde{\tilde{u}},\{\tilde{\tilde{u}}_j\}) = L_1^{(2)}(\tilde{\tilde{u}}+\tilde{\tilde{u}}_1)L_2^{(2)}(\tilde{\tilde{u}}+\tilde{\tilde{u}}_2)\cdots L_{m_1}^{(2)}\left(\tilde{\tilde{u}}+\tilde{\tilde{u}}_{m_1}\right)$$

where the local L-operator is obtained from the r-matrix $r^{(1)}(u)$ and takes the form

$$L_k^{(2)}(\tilde{\tilde{u}}) = \frac{1}{e(\tilde{\tilde{u}})}\begin{pmatrix} b(\tilde{\tilde{u}}) - (b(\tilde{\tilde{u}}) - e(\tilde{\tilde{u}}))e_{11}^k & -c(\tilde{\tilde{u}})e_{21}^k \\ -d(\tilde{\tilde{u}})e_{12}^k & b(\tilde{\tilde{u}}) - (b(\tilde{\tilde{u}}) - e(\tilde{\tilde{u}}))e_{22}^k \end{pmatrix}$$

and

$$T^{(2)^{-1}}(-\tilde{\tilde{u}},\{\tilde{\tilde{u}}_j\}) = L_{m_1}^{(2)^{-1}}\left(-\tilde{\tilde{u}}+\tilde{\tilde{u}}_{m_1}\right)\cdots L_2^{(2)^{-1}}(-\tilde{\tilde{u}}+\tilde{\tilde{u}}_2)L_1^{(2)^{-1}}(-\tilde{\tilde{u}}+\tilde{\tilde{u}}_1).$$

Here we have used the unitarity relation of the $r^{(1)}$-matrix,

$$r_{12}^{(1)}(u)r_{21}^{(1)}(-u) = \frac{\sinh\frac{\gamma(-u+\eta)}{2}\sinh\frac{\gamma(u+\eta)}{2}}{\sinh\frac{\gamma(-u-\eta)}{2}\sinh\frac{\gamma(u-\eta)}{2}}.$$

Therefore, we have the graded Yang–Baxter relation for the nested monodromy matrix,

$$r_{12}^{(1)}(u-v)\overset{1}{T}{}^{(2)}(u,\{u_j\})\overset{2}{T}{}^{(2)}(v,\{u_j\})=\overset{2}{T}{}^{(2)}(v,\{u_j\})\overset{1}{T}{}^{(2)}(u,\{u_j\})r_{12}^{(1)}(u-v). \tag{52}$$

Here $\overset{1}{T}{}^{(2)}(u,\{u_j\}) = T_{13}^{(2)}(u,\{u_j\}), \overset{2}{T}{}^{(2)}(u,\{u_j\}) = T_{23}^{(2)}(u,\{u_j\})$. In order to prove that the $\tau^{(2)}(u)$ in (49) is indeed the boundary transfer matrix with open boundary conditions, we need to prove that $K_{\pm}^{(2)}$ satisfy the nested graded reflection equation and the dual nested graded reflection equation. By a direct calculation, we can prove that the boundary K-matrices $K_{-}^{(2)^{I}}$, $K_{-}^{(2)^{II}}$ and $K_{-}^{(2)^{III}}$ (50) satisfy the nested graded reflection equation:

$$r_{12}^{(1)}(u-v)\overset{1}{K}{}_{-}^{(2)}(u)r_{21}^{(1)}(u+v)\overset{2}{K}{}_{-}^{(2)}(v)=\overset{2}{K}{}_{-}^{(1)}(v)r_{12}^{(1)}(u+v)\overset{1}{K}{}_{-}^{(2)}(u)r_{21}^{(1)}(u-v). \tag{53}$$

We note that the $r^{(1)}$-matrix also satisfies the crossing-unitarity relation,

$$r_{12}^{(1)^{st_1}}(-u+2\eta)\overset{1}{M}{}^{(2)}r_{21}^{(1)^{st_1}}(u)\overset{1}{M}{}^{(2)^{-1}}=\frac{\sinh\frac{\gamma u}{2}\sinh\frac{\gamma(2u-\eta)}{2}}{\sinh\frac{\gamma(u-\eta)}{2}\sinh\frac{\gamma(-u+\eta)}{2}}$$

where $M^{(2)}$ is the crossing matrix: $M^{(2)}=\mathrm{diag}(1,\mathrm{e}^{-\eta\gamma})$. We can also prove that $K_{+}^{(2)^{I}}$, $K_{+}^{(2)^{II}}$ and $K_{+}^{(2)^{III}}$ (51) satisfy the dual nested graded reflection equation:

$$\begin{aligned}&r_{12}^{(1)}(-u+v)\overset{1}{K}{}_{+}^{(2)}(u)\overset{1}{M}{}^{(2)^{-1}}r_{21}^{(1)}(-u-v+2\eta)\overset{1}{M}{}^{(2)}\overset{2}{K}{}_{+}^{(2)}(v)\\&\quad=\overset{1}{M}{}^{(2)}\overset{2}{K}{}_{+}^{(2)}(v)r_{12}^{(1)}(-u-v+2\eta)\overset{1}{M}{}^{(2)^{-1}}\overset{1}{K}{}_{+}^{(2)}(u)r_{21}^{(1)}(-u+v).\end{aligned} \tag{54}$$

One finds that there is an isomorphism between equations (53) and (54) for the case $g=\eta$,

$$K_{+}^{(2)}(u)=M^{(2)}K_{-}^{(2)}(-u+\eta).$$

Thus the nested boundary transfer matrices $\tau^{(2)}(u)$ are shown to constitute a commuting family. We can still use the graded algebraic Bethe ansatz method to find the eigenvalues and eigenvectors of the $\tau^{(2)}(u)$.

3.4. The second nested algebraic Bethe ansatz method

Now, let us use again the graded algebraic Bethe ansatz method to obtain the eigenvalue $\Lambda^{(2)}(u)$ of the nested boundary transfer matrix $\tau^{(2)}(u)$. We write the nested doubled monodromy matrix as

$$\mathcal{T}^{(2)}(u,\{u_i\})=T^{(2)}(u,\{u_i\})K_{-}^{(2)}(u)T^{(2)^{-1}}(-u,\{u_i\})=\begin{pmatrix}\mathcal{A}^{(2)}(u)&\mathcal{B}^{(2)}(u)\\\mathcal{C}^{(2)}(u)&\mathcal{D}^{(2)}(u)\end{pmatrix}. \tag{55}$$

We can show that the $\mathcal{T}^{(2)}(u,\{v_i\})$ in (55) satisfy the equation

$$r_{12}^{(1)}(u-v)\overset{1}{\mathcal{T}}{}^{(2)}(u)r_{21}^{(1)}(u+v)\overset{2}{\mathcal{T}}{}^{(2)}(v)=\overset{2}{\mathcal{T}}{}^{(2)}(v)r_{12}^{(1)}(u+v)\overset{1}{\mathcal{T}}{}^{(2)}(u)r_{21}^{(1)}(u-v). \tag{56}$$

For convenience, we introduce again a transformation

$$\mathcal{D}^{(2)}(u)=\check{\mathcal{D}}^{(2)}(u)-\frac{\sinh\frac{\gamma\eta}{2}\mathrm{e}^{\gamma u}}{\sinh\frac{\gamma(2u-\eta)}{2}}\mathcal{A}^{(2)}(u).$$

One can find the following commutation relations:

$$
\begin{aligned}
\mathcal{A}^{(2)}(u)\mathcal{B}^{(2)}(v) &= \frac{\sinh\frac{\gamma(u-v+\eta)}{2}\sinh\frac{\gamma(u+v)}{2}}{\sinh\frac{\gamma(u-v)}{2}\sinh\frac{\gamma(u+v-\eta)}{2}}\mathcal{B}^{(2)}(v)\mathcal{A}^{(2)}(u)\\
&\quad-\frac{\sinh\gamma v\sinh\frac{\gamma\eta}{2}\,\mathrm{e}^{-\frac{\gamma(u-v)}{2}}}{\sinh\frac{\gamma(u-v)}{2}\sinh\frac{\gamma(2v-\eta)}{2}}\mathcal{B}^{(2)}(u)\mathcal{A}^{(2)}(v)+\frac{\sinh\frac{\gamma\eta}{2}\,\mathrm{e}^{-\frac{\gamma(u+v)}{2}}}{\sinh\frac{\gamma(u+v-\eta)}{2}}\mathcal{B}^{(2)}(u)\check{\mathcal{D}}^{(2)}(v)\\
\check{\mathcal{D}}^{(2)}(u)\mathcal{B}^{(2)}(v) &= \frac{\sinh\frac{\gamma(u-v-\eta)}{2}\sinh\frac{\gamma(u+v-2\eta)}{2}}{\sinh\frac{\gamma(u-v)}{2}\sinh\frac{\gamma(u+v-\eta)}{2}}\mathcal{B}^{(2)}(v)\check{\mathcal{D}}^{(2)}(u)\\
&\quad+\frac{\sinh\frac{\gamma\eta}{2}\sinh\gamma(u-\eta)\,\mathrm{e}^{\frac{\gamma(u-v)}{2}}}{\sinh\frac{\gamma(u-v)}{2}\sinh\frac{\gamma(2u-\eta)}{2}}\mathcal{B}^{(2)}(u)\check{\mathcal{D}}^{(2)}(v)\\
&\quad-\frac{\sinh\gamma v\sinh\gamma(u-\eta)\sinh\frac{\gamma\eta}{2}\,\mathrm{e}^{\frac{\gamma(u+v)}{2}}}{\sinh\frac{\gamma(u+v-\eta)}{2}\sinh\frac{\gamma(2u-\eta)}{2}\sinh\frac{\gamma(2v-\eta)}{2}}\mathcal{B}^{(2)}(u)\mathcal{A}^{(2)}(v).
\end{aligned}
$$

For the local pseudovacuum state $|0\rangle^{(2)}=\otimes_{l=1}^{m_1}|0\rangle_l^{(2)}$, where $|0\rangle_l^{(2)}=(1,0)^T$, we have

$$
\mathcal{C}^{(2)}(\tilde{\tilde{u}})|0\rangle^{(2)}=0\qquad \mathcal{B}^{(2)}(\tilde{\tilde{u}})|0\rangle^{(2)}\neq 0\qquad \mathcal{A}^{(2)}(\tilde{\tilde{u}})|0\rangle^{(2)}=U_1^{(2)}(\tilde{\tilde{u}})|0\rangle^{(2)}
$$

$$
\check{\mathcal{D}}^{(2)}(\tilde{\tilde{u}})|0\rangle^{(2)}=U_2^{(2)}(\tilde{\tilde{u}})\prod_{k=1}^{m_1}\frac{\sinh\frac{\gamma(\tilde{\tilde{u}}+\tilde{\tilde{u}}_k)}{2}\sinh\frac{\gamma(\tilde{\tilde{u}}-\tilde{\tilde{u}}_k)}{2}}{\sinh\frac{\gamma(\tilde{\tilde{u}}+\tilde{\tilde{u}}_k-\eta)}{2}\sinh\frac{\gamma(\tilde{\tilde{u}}-\tilde{\tilde{u}}_k-\eta)}{2}}|0\rangle^{(2)}.
$$

Here $U_1^{(2)}$ and $U_2^{(2)}$ take the following form explicitly,

$$
U_1^{(2)}(\tilde{\tilde{u}})=k_1^{(2)}(\tilde{\tilde{u}})\qquad U_2^{(2)}(\tilde{\tilde{u}})=k_2^{(2)}(\tilde{\tilde{u}})+k_1^{(2)}(\tilde{\tilde{u}})\frac{\sinh\frac{\gamma\eta}{2}\,\mathrm{e}^{\gamma\tilde{\tilde{u}}}}{\sinh\frac{\gamma(2\tilde{\tilde{u}}-\eta)}{2}}
$$

which means

$$
U_1^{(2)}(\tilde{\tilde{u}})=\begin{cases}
\dfrac{\sinh\gamma(\tilde{\tilde{u}}+2)\sinh\frac{\gamma(\tilde{\tilde{u}}+\tilde{\tilde{\xi}}^{I}_{-})}{2}}{\sinh\gamma\tilde{\tilde{u}}\sinh\frac{\gamma\tilde{\tilde{\xi}}^{I}_{-}}{2}}\,\mathrm{e}^{\gamma(2\tilde{\tilde{u}}+3)} & \text{for}\quad\text{Case I}\\[2ex]
\dfrac{\sinh\gamma(\tilde{\tilde{u}}+2)\sinh\frac{\gamma(-\tilde{\tilde{u}}+\tilde{\tilde{\xi}}^{II}_{-})}{2}}{\sinh\gamma\tilde{\tilde{u}}\sinh\frac{\gamma\tilde{\tilde{\xi}}^{II}_{-}}{2}}\,\mathrm{e}^{\gamma(\tilde{\tilde{u}}+3)} & \text{for}\quad\text{Case II}\\[2ex]
\dfrac{\sinh\gamma(\tilde{\tilde{u}}+2)\sinh\frac{\gamma(\tilde{\tilde{u}}+\tilde{\tilde{\xi}}^{III}_{-})}{2}}{\sinh\gamma\tilde{\tilde{u}}\sinh\frac{\gamma(\tilde{\tilde{\xi}}^{III}_{-}+2)}{2}}\,\mathrm{e}^{\gamma(\tilde{\tilde{u}}+2)} & \text{for}\quad\text{Case III}
\end{cases}
$$

$$
U_2^{(2)}(\tilde{\tilde{u}})=\begin{cases}
\dfrac{\sinh\gamma(\tilde{\tilde{u}}+2)\sinh\frac{\gamma(\tilde{\tilde{u}}+\tilde{\tilde{\xi}}^{I}_{-})}{2}}{\sinh\gamma(\tilde{\tilde{u}}+1)\sinh\frac{\gamma\tilde{\tilde{\xi}}^{I}_{-}}{2}}\,\mathrm{e}^{\gamma(2\tilde{\tilde{u}}+2)} & \text{for}\quad\text{Case I}\\[2ex]
\dfrac{\sinh\gamma(\tilde{\tilde{u}}+2)\sinh\frac{\gamma(\tilde{\tilde{u}}+\tilde{\tilde{\xi}}^{II}_{-}+2)}{2}}{\sinh\gamma(\tilde{\tilde{u}}+1)\sinh\frac{\gamma\tilde{\tilde{\xi}}^{II}_{-}}{2}}\,\mathrm{e}^{\gamma(2\tilde{\tilde{u}}+3)} & \text{for}\quad\text{Case II}\\[2ex]
\dfrac{\sinh\gamma(\tilde{\tilde{u}}+2)\sinh\frac{\gamma(\tilde{\tilde{u}}+\tilde{\tilde{\xi}}^{III}_{-})}{2}}{\sinh\gamma(\tilde{\tilde{u}}+1)\sinh\frac{\gamma(\tilde{\tilde{\xi}}^{III}_{-}+2)}{2}}\,\mathrm{e}^{\gamma(\tilde{\tilde{u}}+1)} & \text{for}\quad\text{Case III.}
\end{cases}
$$

The nested boundary transfer matrix takes the form

$$\tau^{(2)}(\tilde{u}) = -k_1^{(2)+}(\tilde{u})\mathcal{A}^{(2)}(\tilde{u}) - k_2^{(2)+}(\tilde{u})\mathcal{D}^{(2)}(\tilde{u})$$
$$= -U_1^{(2)+}(\tilde{u})\mathcal{A}^{(2)}(\tilde{u}) - U_2^{(2)+}(\tilde{u})\check{\mathcal{D}}^{(2)}(\tilde{u}) \tag{57}$$

where we denote

$$U_1^{(2)+}(\tilde{u}) = k_1^{(2)+}(\tilde{u}) - k_2^{(1)+}(\tilde{u})\frac{\sinh\frac{\gamma\eta}{2}\,\mathrm{e}^{\gamma\tilde{u}}}{\sinh\frac{\gamma(2\tilde{u}-\eta)}{2}} \qquad U_2^{(2)+}(\tilde{u}) = k_2^{(2)+}(\tilde{u}).$$

This means that

$$U_1^{(2)+}(\tilde{u}) = \begin{cases} \dfrac{\sinh\gamma(\tilde{u}+2)\sinh\frac{\gamma\left(-\tilde{u}+\tilde{\xi}_+^{I}-2\right)}{2}}{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma\tilde{\xi}_+^{I}}{2}}\,\mathrm{e}^{-\gamma(2\tilde{u}+5)} & \text{for Case I} \\ \dfrac{\sinh\gamma(\tilde{u}+2)\sinh\frac{\gamma\left(\tilde{u}+\tilde{\xi}_+^{II}\right)}{2}}{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma\tilde{\xi}_+^{II}}{2}}\,\mathrm{e}^{-\gamma(\tilde{u}+4)} & \text{for Case II} \\ \dfrac{\sinh\gamma(\tilde{u}+2)\sinh\frac{\gamma\left(-\tilde{u}+\tilde{\xi}_+^{III}-2\right)}{2}}{\sinh\gamma(\tilde{u}+1)\sinh\frac{\gamma\tilde{\xi}_+^{III}}{2}}\,\mathrm{e}^{-\gamma(\tilde{u}+3)} & \text{for Case III} \end{cases}$$

$$U_2^{(2)+}(\tilde{u}) = \begin{cases} \dfrac{\sinh\frac{\gamma\left(-\tilde{u}+\tilde{\xi}_+^{I}-2\right)}{2}}{\sinh\frac{\gamma\tilde{\xi}_+^{I}}{2}}\,\mathrm{e}^{-\gamma(2\tilde{u}+4)} & \text{for Case I} \\ \dfrac{\sinh\frac{\gamma\left(-\tilde{u}+\tilde{\xi}_+^{II}-2\right)}{2}}{\sinh\frac{\gamma\tilde{\xi}_+^{II}}{2}}\,\mathrm{e}^{-\gamma(2\tilde{u}+4)} & \text{for Case II} \\ \dfrac{\sinh\frac{\gamma\left(-\tilde{u}+\tilde{\xi}_+^{III}-2\right)}{2}}{\sinh\frac{\gamma\tilde{\xi}_+^{III}}{2}}\,\mathrm{e}^{-\gamma(\tilde{u}+2)} & \text{for Case III.} \end{cases}$$

Following the standard algebraic Bethe ansatz method, acting the nested boundary transfer matrix $\tau^{(2)}(\tilde{u})$ (57) on the states

$$\mathcal{C}\big(\tilde{\tilde{u}}_1^{(2)}\big)\mathcal{C}\big(\tilde{\tilde{u}}_2^{(2)}\big)\cdots\mathcal{C}\big(\tilde{\tilde{u}}_{m_1}^{(2)}\big)|0\rangle^{(2)}$$

we find the eigenvalue of $\tau^{(2)}(\tilde{u})$ as follows,

$$\Lambda^{(2)}(\tilde{u}) = -U_1^{(2)+}(\tilde{u})U_1^{(2)}(\tilde{u})\prod_{l=1}^{m_2}\frac{\sinh\frac{\gamma\left(\tilde{u}-\tilde{\tilde{u}}_l^{(2)}+\eta\right)}{2}\sinh\frac{\gamma\left(\tilde{u}+\tilde{\tilde{u}}_l^{(2)}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}-\tilde{\tilde{u}}_l^{(2)}\right)}{2}\sinh\frac{\gamma\left(\tilde{u}+\tilde{\tilde{u}}_l^{(2)}-\eta\right)}{2}} - U_2^{(2)+}(\tilde{u})U_2^{(2)}(\tilde{u})$$
$$\times\prod_{k=1}^{m_1}\frac{\sinh\frac{\gamma\left(\tilde{u}+\tilde{\tilde{u}}_k^{(1)}\right)}{2}\sinh\frac{\gamma\left(\tilde{u}-\tilde{\tilde{u}}_k^{(1)}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}+\tilde{\tilde{u}}_k^{(1)}-\eta\right)}{2}\sinh\frac{\gamma\left(\tilde{u}-\tilde{\tilde{u}}_k^{(1)}-\eta\right)}{2}}\prod_{l=1}^{m_2}\frac{\sinh\frac{\gamma\left(\tilde{u}-\tilde{\tilde{u}}_l^{(2)}\right)}{2}\sinh\frac{\gamma\left(\tilde{u}+\tilde{\tilde{u}}_l^{(2)}-2\eta\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}-\tilde{\tilde{u}}_l^{(2)}\right)}{2}\sinh\frac{\gamma\left(\tilde{u}+\tilde{\tilde{u}}_l^{(2)}-\eta\right)}{2}} \tag{58}$$

provided the parameters $\left\{\tilde{\tilde{u}}_l^{(2)}\right\}$ satisfy the following Bethe ansatz equation:

$$\frac{U_2^{(2)+}\left(\tilde{\tilde{u}}_l^{(2)}\right)U_2^{(2)}\left(\tilde{\tilde{u}}_l^{(2)}\right)}{U_1^{(2)+}\left(\tilde{\tilde{u}}_l^{(2)}\right)U_1^{(2)}\left(\tilde{\tilde{u}}_l^{(2)}\right)}\frac{\sinh\gamma\left(\tilde{\tilde{u}}_l^{(2)}-\eta\right)}{\sinh\gamma\tilde{\tilde{u}}_l^{(2)}}\prod_{k=1}^{m_1}\frac{\sinh\frac{\gamma\left(\tilde{\tilde{u}}_l^{(2)}+\tilde{\tilde{u}}_k^{(1)}\right)}{2}\sinh\frac{\gamma\left(\tilde{\tilde{u}}_l^{(2)}-\tilde{\tilde{u}}_k^{(1)}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{\tilde{u}}_l^{(2)}+\tilde{\tilde{u}}_k^{(1)}-\eta\right)}{2}\sinh\frac{\gamma\left(\tilde{\tilde{u}}_l^{(2)}-\tilde{\tilde{u}}_k^{(1)}-\eta\right)}{2}}$$

$$= \prod_{\substack{p=1 \\ p\neq l}}^{m_2} \frac{\sinh\frac{\gamma\left(\tilde{\tilde{u}}_l^{(2)}-\tilde{\tilde{u}}_p^{(2)}+\eta\right)}{2}\sinh\frac{\gamma\left(\tilde{\tilde{u}}_l^{(2)}+\tilde{\tilde{u}}_p^{(2)}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{\tilde{u}}_l^{(2)}-\tilde{\tilde{u}}_p^{(2)}-\eta\right)}{2}\sinh\frac{\gamma\left(\tilde{\tilde{u}}_l^{(2)}+\tilde{\tilde{u}}_p^{(2)}-2\eta\right)}{2}}. \tag{59}$$

By (48) and (58), we have

$$\prod_{j=1}^{n} \frac{\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}+\tilde{u}_j+\eta\right)}{2}\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}-\tilde{u}_j+\eta\right)}{2}}{\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}+\tilde{u}_j\right)}{2}\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}-\tilde{u}_j\right)}{2}}$$

$$= \frac{U_1^{(2)^+}\big(\tilde{\tilde{u}}_k^{(1)}\big)U_1^{(2)}\big(\tilde{\tilde{u}}_k^{(1)}\big)}{U_1^{(1)^+}\big(\tilde{\tilde{u}}_k^{(1)}\big)U_1^{(1)}\big(\tilde{\tilde{u}}_k^{(1)}\big)} \prod_{l=1}^{m_2} \frac{\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}-\tilde{\tilde{u}}_l^{(2)}+\eta\right)}{2}\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}+\tilde{\tilde{u}}_l^{(2)}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}-\tilde{\tilde{u}}_l^{(2)}\right)}{2}\sinh\frac{\gamma\left(\tilde{\tilde{u}}_k^{(1)}+\tilde{\tilde{u}}_l^{(2)}-\eta\right)}{2}}. \tag{60}$$

By (30) and (47), we have

$$\frac{U_1^{+}(\tilde{u}_j)U_1(\tilde{u}_j)}{U_1^{(1)^+}(\tilde{u}_j)U_1^{(1)}(\tilde{u}_j)} \frac{\sinh\gamma(\tilde{u}_j+1)}{\sinh\gamma(\tilde{u}_j-1)} \left(\frac{\sinh\frac{\gamma(\tilde{u}_j-1)}{2}}{\sinh\frac{\gamma(\tilde{u}_j+1)}{2}}\right)^{2L} \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{\sinh\frac{\gamma(\tilde{u}_j+\tilde{u}_i+2)}{2}\sinh\frac{\gamma(\tilde{u}_j-\tilde{u}_i+2)}{2}}{\sinh\frac{\gamma(\tilde{u}_j+\tilde{u}_i-2)}{2}\sinh\frac{\gamma(\tilde{u}_j-\tilde{u}_i-2)}{2}}$$

$$= \prod_{k=1}^{m_1} \frac{\sinh\frac{\gamma\left(\tilde{u}_j-\tilde{\tilde{u}}_k^{(1)}+2\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{\tilde{u}}_k^{(1)}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_j-\tilde{\tilde{u}}_k^{(1)}\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{\tilde{u}}_k^{(1)}-2\right)}{2}}. \tag{61}$$

After a shift of the parameters $\tilde{\tilde{u}}_k^{(1)} = \tilde{u}_k^{(1)}+1$, $\tilde{\tilde{u}}_l^{(2)} = \tilde{u}_l^{(2)}-1$ and $\tilde{\tilde{\xi}}_- = \tilde{\xi}_- - 1$, $\tilde{\tilde{\xi}}_+ = \tilde{\xi}_+ + 1$, the Bethe ansatz equations (61), (60) and (59) may be rewritten as follows:

$$\frac{U_1^{+}(\tilde{u}_j)U_1(\tilde{u}_j)}{U_1^{(1)^+}(\tilde{u}_j)U_1^{(1)}(\tilde{u}_j)} \frac{\sinh\gamma(\tilde{u}_j+1)}{\sinh\gamma(\tilde{u}_j-1)} \left(\frac{\sinh\frac{\gamma(\tilde{u}_j-1)}{2}}{\sinh\frac{\gamma(\tilde{u}_j+1)}{2}}\right)^{2L} \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{\sinh\frac{\gamma(\tilde{u}_j+\tilde{u}_i+2)}{2}\sinh\frac{\gamma(\tilde{u}_j-\tilde{u}_i+2)}{2}}{\sinh\frac{\gamma(\tilde{u}_j+\tilde{u}_i-2)}{2}\sinh\frac{\gamma(\tilde{u}_j-\tilde{u}_i-2)}{2}}$$

$$= \prod_{k=1}^{m_1} \frac{\sinh\frac{\gamma\left(\tilde{u}_j-\tilde{u}_k^{(1)}+1\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{u}_k^{(1)}+1\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_j-\tilde{u}_k^{(1)}-1\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{u}_k^{(1)}-1\right)}{2}} \tag{62}$$

$$\prod_{j=1}^{n} \frac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{u}_j-1\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}-\tilde{u}_j-1\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{u}_j+1\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}-\tilde{u}_j+1\right)}{2}}$$

$$= \frac{U_1^{(2)^+}\big(\tilde{u}_k^{(1)}+1\big)U_1^{(2)}\big(\tilde{u}_k^{(1)}+1\big)}{U_1^{(1)^+}\big(\tilde{u}_k^{(1)}+1\big)U_1^{(1)}\big(\tilde{u}_k^{(1)}+1\big)} \prod_{l=1}^{m_2} \frac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}-\tilde{u}_l^{(2)}\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{u}_l^{(2)}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}-\tilde{u}_l^{(2)}+2\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{u}_l^{(2)}+2\right)}{2}} \tag{63}$$

$$\frac{U_2^{(2)^+}\big(\tilde{u}_l^{(2)}-1\big)U_2^{(2)}\big(\tilde{u}_l^{(2)}-1\big)}{U_1^{(2)^+}\big(\tilde{u}_l^{(2)}-1\big)U_1^{(2)}\big(\tilde{u}_l^{(2)}-1\big)} \frac{\sinh\gamma\big(\tilde{u}_l^{(2)}+1\big)}{\sinh\gamma\big(\tilde{u}_l^{(2)}-1\big)} \prod_{k=1}^{m_1} \frac{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{u}_k^{(1)}\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}-\tilde{u}_k^{(1)}-2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{u}_k^{(1)}+2\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}-\tilde{u}_k^{(1)}\right)}{2}}$$

$$= \prod_{\substack{p=1\\ p\neq l}}^{m_2} \frac{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}-\tilde{u}_p^{(2)}-2\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{u}_p^{(2)}-2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}-\tilde{u}_p^{(2)}+2\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{u}_p^{(2)}+2\right)}{2}}. \tag{64}$$

The Bethe ansatz equations for all the nine cases are as follows:

$$F(\tilde{u}_j)\left(\frac{\sinh\frac{\gamma(\tilde{u}_j-1)}{2}}{\sinh\frac{\gamma(\tilde{u}_j+1)}{2}}\right)^{2L}\prod_{\substack{i=1\\ i\neq j}}^{n}\frac{\sinh\frac{\gamma(\tilde{u}_j+\tilde{u}_i+2)}{2}\sinh\frac{\gamma(\tilde{u}_j-\tilde{u}_i+2)}{2}}{\sinh\frac{\gamma(\tilde{u}_j+\tilde{u}_i-2)}{2}\sinh\frac{\gamma(\tilde{u}_j-\tilde{u}_i-2)}{2}}$$
$$=\prod_{k=1}^{m_1}\frac{\sinh\frac{\gamma\left(\tilde{u}_j-\tilde{u}_k^{(1)}+1\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{u}_k^{(1)}+1\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_j-\tilde{u}_k^{(1)}-1\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{u}_k^{(1)}-1\right)}{2}} \tag{65}$$

$$G\left(\tilde{u}_k^{(1)}\right)\prod_{j=1}^{n}\frac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{u}_j-1\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}-\tilde{u}_j-1\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{u}_j+1\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}-\tilde{u}_j+1\right)}{2}}=\prod_{l=1}^{m_2}\frac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}-\tilde{u}_l^{(2)}\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{u}_l^{(2)}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}-\tilde{u}_l^{(2)}+2\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{u}_l^{(2)}+2\right)}{2}} \tag{66}$$

$$K\left(\tilde{u}_l^{(2)}\right)\prod_{k=1}^{m_1}\frac{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{u}_k^{(1)}\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}-\tilde{u}_k^{(1)}-2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{u}_k^{(1)}+2\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}-\tilde{u}_k^{(1)}\right)}{2}}=\prod_{\substack{p=1\\ p\neq l}}^{m_2}\frac{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}-\tilde{u}_p^{(2)}-2\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{u}_p^{(2)}-2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}-\tilde{u}_p^{(2)}+2\right)}{2}\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{u}_p^{(2)}+2\right)}{2}} \tag{67}$$

with

$$F(\tilde{u}_j)=\begin{cases}\dfrac{\sinh\frac{\gamma\left(-\tilde{u}_j+\tilde{\xi}_+^{I}-2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{\xi}_+^{I}-2\right)}{2}} & \text{(Case (i))}\\[2ex] \dfrac{\sinh\frac{\gamma(\tilde{u}_j+\xi_-^{II})}{2}}{\sinh\frac{\gamma(\tilde{u}_j-\xi_-^{II})}{2}}\dfrac{\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{\xi}_+^{II}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_j-\tilde{\xi}_+^{II}\right)}{2}} & \text{(Case (ii))}\\[2ex] 1 & \text{(Case (iii))}\\[1ex] \dfrac{\sinh\frac{\gamma\left(-\tilde{u}_j+\tilde{\xi}_+^{II}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{\xi}_+^{II}\right)}{2}} & \text{(Case (iv))}\\[2ex] \dfrac{\sinh\frac{\gamma(\tilde{u}_j+\xi_-^{II})}{2}}{\sinh\frac{\gamma(\tilde{u}_j-\xi_-^{II})}{2}}\dfrac{\sinh\frac{\gamma\left(\tilde{u}_j-\tilde{\xi}_+^{I}+2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{\xi}_+^{I}-2\right)}{2}} & \text{(Case (v))}\\[2ex] \mathrm{e}^{-\gamma\tilde{u}_j} & \text{(Case (vi))}\\[1ex] \dfrac{\sinh\frac{\gamma\left(-\tilde{u}_j+\tilde{\xi}_+^{I}-2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{\xi}_+^{I}-2\right)}{2}}\mathrm{e}^{\gamma\tilde{u}_j} & \text{(Case (vii))}\\[2ex] \dfrac{\sinh\frac{\gamma(\tilde{u}_j+\xi_-^{II})}{2}}{\sinh\frac{\gamma(-\tilde{u}_j+\xi_-^{II})}{2}}\mathrm{e}^{-\gamma\tilde{u}_j} & \text{(Case (viii))}\\[2ex] \dfrac{\sinh\frac{\gamma\left(-\tilde{u}_j+\tilde{\xi}_+^{II}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_j+\tilde{\xi}_+^{II}\right)}{2}}\mathrm{e}^{\gamma\tilde{u}_j} & \text{(Case (ix))}\end{cases} \tag{68}$$

$$
G\left(\tilde{u}_k^{(1)}\right)=\begin{cases}
\dfrac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_-^{I}+2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_-^{I}\right)}{2}}\,\dfrac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_+^{I}\right)}{2}}{\sinh\frac{\gamma\left(-\tilde{u}_k^{(1)}+\tilde{\xi}_+^{I}-2\right)}{2}} & \text{(Case (i))}\\
1 & \text{(Case (ii))}\\
\dfrac{\sinh\frac{\gamma\left(-\tilde{u}_k^{(1)}+\tilde{\xi}_-^{III}-2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_-^{III}\right)}{2}}\,\dfrac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_+^{III}\right)}{2}}{\sinh\frac{\gamma\left(-\tilde{u}_k^{(1)}+\tilde{\xi}_+^{III}-2\right)}{2}} & \text{(Case (iii))}\\
\dfrac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_-^{I}+2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_-^{I}\right)}{2}}\,\mathrm{e}^{-\gamma\left(\tilde{u}_k^{(1)}+1\right)} & \text{(Case (iv))}\\
\dfrac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_+^{I}\right)}{2}}{\sinh\frac{\gamma\left(-\tilde{u}_k^{(1)}+\tilde{\xi}_+^{I}-2\right)}{2}}\,\mathrm{e}^{\gamma\left(\tilde{u}_k^{(1)}+1\right)} & \text{(Case (v))}\\
\dfrac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_-^{I}+2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_-^{I}\right)}{2}}\,\dfrac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_+^{III}\right)}{2}}{\sinh\frac{\gamma\left(-\tilde{u}_k^{(1)}+\tilde{\xi}_+^{III}-2\right)}{2}} & \text{(Case (vi))}\\
\dfrac{\sinh\frac{\gamma\left(-\tilde{u}_k^{(1)}+\tilde{\xi}_-^{III}-2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_-^{III}\right)}{2}}\,\dfrac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_+^{I}\right)}{2}}{\sinh\frac{\gamma\left(-\tilde{u}_k^{(1)}+\tilde{\xi}_+^{I}-2\right)}{2}} & \text{(Case (vii))}\\
\dfrac{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_+^{III}\right)}{2}}{\sinh\frac{\gamma\left(-\tilde{u}_k^{(1)}+\tilde{\xi}_+^{III}-2\right)}{2}}\,\mathrm{e}^{\gamma\left(\tilde{u}_k^{(1)}+1\right)} & \text{(Case (viii))}\\
\dfrac{\sinh\frac{\gamma\left(-\tilde{u}_k^{(1)}+\tilde{\xi}_-^{III}-2\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_k^{(1)}+\tilde{\xi}_-^{III}\right)}{2}}\,\mathrm{e}^{-\gamma\left(\tilde{u}_k^{(1)}+1\right)} & \text{(Case (ix))}
\end{cases}
\tag{69}
$$

$$
K\left(\tilde{u}_l^{(2)}\right)=\begin{cases}
1 & \text{(Case (i))}\\
\dfrac{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{\xi}_-^{II}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}-\tilde{\xi}_-^{II}\right)}{2}}\,\dfrac{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}-\tilde{\xi}_+^{II}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{\xi}_+^{II}\right)}{2}} & \text{(Case (ii))}\\
1 & \text{(Case (iii))}\\
\dfrac{\sinh\frac{\gamma\left(-\tilde{u}_l^{(2)}+\tilde{\xi}_+^{II}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{\xi}_+^{II}\right)}{2}}\,\mathrm{e}^{-\gamma\tilde{u}_l^{(2)}} & \text{(Case (iv))}\\
\dfrac{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{\xi}_-^{II}\right)}{2}}{\sinh\frac{\gamma\left(-\tilde{u}_l^{(2)}+\tilde{\xi}_-^{II}\right)}{2}}\,\mathrm{e}^{\gamma\tilde{u}_l^{(2)}} & \text{(Case (v))}\\
1 & \text{(Case (vi))}\\
1 & \text{(Case (vii))}\\
\dfrac{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{\xi}_-^{II}\right)}{2}}{\sinh\frac{\gamma\left(-\tilde{u}_l^{(2)}+\tilde{\xi}_-^{II}\right)}{2}}\,\mathrm{e}^{\gamma\tilde{u}_l^{(2)}} & \text{(Case (viii))}\\
\dfrac{\sinh\frac{\gamma\left(-\tilde{u}_l^{(2)}+\tilde{\xi}_+^{II}\right)}{2}}{\sinh\frac{\gamma\left(\tilde{u}_l^{(2)}+\tilde{\xi}_+^{II}\right)}{2}}\,\mathrm{e}^{-\gamma\tilde{u}_l^{(2)}} & \text{(Case (ix))}
\end{cases}
\tag{70}
$$

$$E = -\sum_{j=1}^{n} \frac{4}{\sinh \frac{\gamma(u_j-1)}{2} \sinh \frac{\gamma(u_j+1)}{2}} \tag{71}$$

(modulo an unimportant additive constant, which we drop).

4. Conclusion

In conclusion, we have studied integrable open boundary conditions of a one-dimensional q-deformed $U_q(gl(2|2))$ extended Hubbard model. The eigenvalues of the Hamiltonian are discussed through the algebraic Bethe ansatz, by combining Sklyanin's algebraic Bethe ansatz for the open boundary XXZ chain and the nested construction of algebraic Bethe ansatz for the $U_q(gl(2|2))$ extended Hubbard model. The Bethe ansatz equations for the nine classes of integrable open boundary cases are obtained by means of the algebraic Bethe ansatz approach. More specific future works are planned as follows: (i) studying low energy behaviour and physical properties of the corresponding systems based on an analysis of the Bethe ansatz equations which we have obtained in this paper, including, investigating the ground state structure, computing the finite-size corrections to the low-lying energies, and calculating thermodynamic equilibrium properties; (ii) developing some traditional mathematical methods such as the Wiener–Hopf technique to solve the special kind of integral equations arising from the thermodynamic Bethe ansatz equations, calculating the boundary susceptibility and the low temperature limit for open boundary conditions; (iii) establishing the finite-size spectrum analytically and drawing boundary critical behaviour properties together with the boundary conformal field theory techniques in cases of open boundary conditions.

Acknowledgment

This work is supported by the Australian Research Council.

References

[1] Bethe H A 1931 *Z. Phys.* **71** 205
[2] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[3] Lieb E H and Wu F-Y 1968 *Phys. Rev. Lett.* **20** 1445
[4] Baxter R J 1972 *Ann. Phys., NY* **70** 193
[5] Uimin G 1970 *JETP Lett.* **12** 225
[6] Lai J K 1974 *J. Math. Phys.* **15** 1675
[7] Sutherland B 1975 *Phys. Rev.* B **12** 3795
[8] Sklyanin E K, Takhtajan L A and Faddeev L D 1980 *Theor. Math. Phys.* **40** 688
[9] Anderson P W 1987 *Science* **235** 1196
[10] Zhang F C and Rice T M 1988 *Phys. Rev.* B **37** 3759
[11] Shastry B S 1986 *Phys. Rev. Lett.* **56** 2453
Shastry B S 1988 *J. Stat. Phys.* **50** 57
[12] Woynarovich F 1989 *J. Phys. A: Math. Gen.* **22** 4243
[13] Frahm H and Korepin V E 1991 *Phys. Rev.* B **43** 5653
[14] Ramos P B and Martins M J 1997 *J. Phys. A: Math. Gen.* **30** L195
[15] Bares P A and Blatter G 1990 *Phys. Rev.* B **64** 2567
[16] Sarkar S 1990 *J. Phys. A: Math. Gen.* **23** L409
Sarkar S 1991 *J. Phys. A: Math. Gen.* **24** 1137
[17] Essler F H L and Korepin V E 1992 *Phys. Rev.* B **46** 9147
[18] Klümper A, Schadschneider A and Zittartz J 1991 *J. Phys. A: Math. Gen.* **24** L955
[19] Bariev R Z 1994 *J. Phys. A: Math. Gen.* **27** 3381

[20] Foerster A and Karowski M 1993 *Nucl. Phys.* B **396** 611
Foerster A and Karowski M 1993 *Nucl. Phys.* B **408** 512
[21] Essler F H L, Korepin V E and Schoutens K 1992 *Phys. Rev. Lett.* **68** 2960
Essler F H L, Korepin V E and Schoutens K 1993 *Phys. Rev. Lett.* **70** 73
[22] Bracken A J, Gould M D, Links J R and Zhang Y-Z 1995 *Phys. Rev. Lett.* **74** 2768
[23] Bedürftig G and Frahm H 1995 *J. Phys. A: Math. Gen.* **28** 4453
[24] Pfannmüller M P and Frahm H 1996 *Nucl. Phys.* B **479** 575
[25] Ramos P B and Martins M J 1996 *Nucl. Phys.* B **474** 678
[26] Essler F H L, Korepin V E and Schoutens K 1994 *Int. J. Mod. Phys.* B **8** 3205
[27] Schoutens K 1994 *Nucl. Phys.* B **413** 675
[28] Essler F H L and Korepin V E 1994 *Int. J. Mod. Phys.* B **8** 3243
[29] Shlottmann P 1997 *Int. J. Mod. Phys.* B **11** 355
[30] Ambjøm J, Karakhanyan D, Mirumyan M and Sedrakyan A 2001 *Nucl. Phys.* B **599** 547
[31] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
[32] Cherednik I V 1984 *Theor. Math. Phys.* **61** 977
[33] Alcaraz F C, Barber M N, Batchelor M T, Baxter R J and Quispel G 1987 *J. Phys. A: Math. Gen.* **20** 6397
[34] Mezincescu L and Nepomechie R 1991 *J. Phys. A: Math. Gen.* **24** L17
Mezincescu L and Nepomechie R 1991 *Int. J. Mod. Phys.* A **6** 5231
[35] Zhou H-Q 1996 *Phys. Rev.* B **54** 41
Zhou H-Q 1996 *Phys. Rev.* B **53** 5089
[36] Shiroishi M and Wadati M 1997 *J. Phys. Soc. Japan.* **66** 1
Shiroishi M and Wadati M 1997 *J. Phys. Soc. Japan.* **66** 2288
[37] Asakawa H and Suzuki M 1996 *J. Phys. A: Math. Gen.* **29** 225
[38] Tsuchiya O and Yamamoto T 1997 *J. Phys. Soc. Japan.* **66** 1950
[39] Bedürftig G and Frahm H 1997 *J. Phys. A: Math. Gen.* **30** 4413
[40] Yue R-H and Deguch T 1997 *J. Phys. A: Math. Gen.* **30** 8129
[41] Guan X-W 2000 *J. Phys. A: Math. Gen.* **33** 5391
[42] González-Ruiz A 1994 *Nucl. Phys.* B **424** 468
[43] Essler F H L 1996 *J. Phys. A: Math. Gen.* **29** 6183
[44] Asakawa H and Suzuki M 1997 *Int. J. Mod. Phys.* B **11** 1137
[45] Essler F H L and Frahm H 1997 *Phys. Rev.* B **56** 6631
[46] Bariev R Z, Klümper A, Schadschneider A and Zittartz J 1995 *Z. Phys.* B **96** 395
Bariev R Z, Klümper A, Schadschneider A and Zittartz J 1995 *J. Phys. A: Math. Gen.* **28** 2437
[47] Bracken A J, Ge X-Y, Zhang Y-Z and Zhou H-Q 1998 *Nucl. Phys.* B **516** 588
[48] Zhou H-Q and Gould M D 1999 *Phys. Lett.* A **251** 279
Zhou H-Q, Ge X-Y, Links J and Gould M D 1999 *Nucl. Phys.* B **546** 779
Zhou H-Q, Ge X-Y, Links J and Gould M D 2000 *Phys. Rev.* B **62** 4906
[49] Ge X-Y, Gould M D, Links J and Zhou H-Q 2001 *J. Phys. A: Math. Gen.* **34** 8543
[50] Fan H, Wadati M and Wang X-M 2000 *Phys. Rev.* B **61** 3450
Fan H, Wadati M and Yue R-H 2000 *J. Phys. A: Math. Gen.* **33** 6187
[51] Ge X-Y 1999 *Mod. Phys. Lett.* B **13** 499